DEFORMATIONS OF HOMOTOPY THEORIES VIA ALGEBRAIC THEORIES

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois Urbana-Champaign, 2021

Urbana, Illinois

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Abstract

We develop a homotopical variant of the classic notion of an algebraic theory as a tool for producing deformations of homotopy theories. From this, we extract a framework for constructing and reasoning with obstruction theories and spectral sequences that compute homotopical data starting with purely algebraic data. We investigate the algebra necessary to apply this to examples of interest, such as to $\mathbb{E}_\infty$ rings with good theories of power operations. As an application, we give some tools for working with $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum of height h, and use these to produce new $\mathbb{E}_\infty$ complex orientations at heights $h \leq 2$.

Acknowledgements

I would first like to thank my advisor, Charles Rezk, without whom this thesis would not be possible. I would then like to thank the mathematics community at the University of Illinois, for making graduate school an exceedingly enjoyable experience. In particular, my time in graduate school would have been much less pleasant were it not for the friendship of many people, among them Daniel Carmody, Ravi Donepudi, Yong Escobar, Jesse Huang, Abhra Kundu, Xiao Li, Ruth Luo, Tsutomu Okano, and Chieu-Minh Tran. Finally, I would like to thank my parents for their love and support.

Contents

CHAPTER 1

Introduction

1.1. Brief summary

Let R be an $\mathbb{E}_\infty$ ring spectrum. Then there is a theory of *R-power operations*: for any $\mathbb{E}_\infty$ algebra A over R, there are natural maps

$$R_b(\mathbb{S}^a)^{\otimes n}_{\mathrm{h}\Sigma_n} \times \pi_a A \to \pi_b A$$

refining the n'th power map. As with all natural operations, these are immediately applicable to nonexistence theorems. For example, a map $\phi\colon \pi_* A \to \pi_* B$ that fails to be compatible with these operations must also fail to refine to a map $A \to B$ of $\mathbb{E}_\infty$ algebras over R. The converse is false, making existence theorems more subtle. As a general heuristic, by understanding the global structure of these operations, one can start to quantify the failure of the converse by introducing a suitable *obstruction theory*.

To illustrate this heuristic we consider a simpler context. If M and N are R-modules, then there are universal coefficient and Künneth spectral sequences

$$\mathrm{Ext}^{p+q}_{R_*}(M_*, N_{*+p}) \Rightarrow \pi_{*-q}\mathcal{M}\mathrm{od}_R(M,N),$$
$$\mathrm{Tor}^{R_*}_{p+q}(M_*, N_{*-p}) \Rightarrow \pi_{*+q} M \otimes_R N.$$

In this example, it is the algebra of R_*-modules that captures the global structure of operations on the homotopy groups of R-modules. The existence of these spectral sequences may be viewed as saying that homotopy theory of R-modules is, in some sense, approximated by the homological algebra of R_*-modules.

The purpose of this thesis is to describe a certain conceptual ∞-categorical framework in which this heuristic is made precise, and to give applications. Central to our approach is the notion of an *algebraic theory*. Building on insights of Hopkins-Lurie [**HL17**] and Pstrągowski [**Pst17**], we introduce a certain homotopical refinement of the classic notion of an algebraic theory, which we call *loop theories*. If $\mathcal{P}$ is a loop theory, then:

(1) There is a category $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ of models of the theory $\mathcal{P}$ that respect the additional homotopical structure present;
(2) There is a category $\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}}$ of set-valued models of the homotopy category $\mathrm{h}\mathcal{P}$, and a natural "homotopy groups" functor $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}}$.

For example, when $\mathcal{P} = \mathcal{M}\mathrm{od}_R^{\mathrm{free}}$, the category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ recovers the category of R-modules, and $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$ is equivalent to the ordinary category of R_*-modules; when $\mathcal{P} = \mathcal{C}\mathrm{Alg}_{H\mathbb{F}_p}^{\mathrm{free}}$, the category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ recovers the category of $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$, and $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$ is equivalent to a category of $\mathbb{F}_{p*}$-rings with Dyer-Lashof operations. In general, $\mathcal{M} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ whenever $\mathcal{M}$ is a homotopy theory with a good subcategory $\mathcal{P} \subset \mathcal{M}$ of free objects, and $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$ is then an algebraic approximation to $\mathcal{M}$. However the key point is the third:

(3) The category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ fits into a span

$$\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} \longleftarrow \mathcal{M}\mathrm{odel}_{\mathcal{P}} \longrightarrow \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} ,$$

and behaves as a *deformation* with generic fiber $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ and special fiber $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$.

Here, $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$ is equivalent to Quillen's homotopy theory of simplicial set-valued models of the algebraic theory $\mathrm{h}\mathcal{P}$. The deformation theory available in this context then gives concrete computational tools, in the form of obstruction-theoretic machinery, for understanding the homotopy theory of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ starting with the algebra of $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$. For example, there is an obstruction theory computing maps in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ with obstruction groups given in terms of the Quillen cohomology of objects of $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$. This story is developed in Chapter 2.

The original motivation for this thesis was to develop tools for working with some particular objects arising in chromatic homotopy theory, the *Lubin-Tate spectra*, or Morava E-theories. These are cohomology theories constructed by Morava from the deformation theory of formal groups, and are $\mathbb{E}_\infty$ ring spectra by work of Goerss-Hopkins-Miller [**GH04**] [**GH05**]. Power operations for $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum of height h were introduced by Ando [**And95**], and their global structure is now well-understood due to work of Ando, Hopkins, Strickland, and Rezk [**Str98**] [**AHS04**] [**Rez09**]; in short, these operations are governed by the deformation theory of *isogenies* of formal groups. This leads to the following question: if E is a Lubin-Tate spectrum of height h, what does the algebra of E-power operations tell us about the homotopy theory of $K(h)$-local $\mathbb{E}_\infty$ algebras over E?

It is exactly questions of this sort that our framework seeks to address. In this particular example, there are at least two things to note:

(1) E-power operations are inherently *infinitary*, and the framework must accomodate this;
(2) Given the general obstruction theory, one must still compute its obstruction groups.

The infinitary nature of E-power operations arises from the $K(h)$-local condition, which forces a certain completeness condition on homotopy groups. This is incorporated by allowing our theories to themselves be infinitary theories. In addition to handling a number of complications related to completions, this turns out to simplify the general story, in many cases amounting to just a removal of what would have otherwise been unused finiteness assumptions.

We turn to the second point in Chapter 3, where we develop some of the general algebra necessary for computing in various categories of the form $\mathrm{Model}_{\mathrm{h}\mathcal{P}}$. Here, the two motivating examples are the theories of power operations for Lubin-Tate spectra and of power operations for $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$; these share a number of properties, and this chapter sets up the general context in which they may be put on the same footing. A key concept here is that of an *algebra over a theory*, due to Freyd [**Fre66**] and Wraith [**Wra71**], and its specialization to the notion of a *plethory*, generalizing those studied by Tall-Wraith [**TW70**] and Borger-Wieland [**BW05**], of which both of these theories of power operations are an example. In this context many Quillen cohomology computations may be split into a classic part, such as of ordinary André-Quillen cohomology, and a purely linear part, where all the methods of homological algebra apply. These linear contexts turn out to frequently admit *Koszul resolutions*, and to access this we also develop the theory of Koszul algebras in the necessary generality.

We end in Chapter 4 with two extended examples, applying all of the general machinery of Chapter 2 and Chapter 3 to the study of $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$ and to the study of $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum of height h. In the former case, we both recover and strengthen various classic obstruction theories available in this context. The latter case returns to our original motivation; here the tools we obtain are new, and turn out to be quite pleasant at heights $h \leq 2$: many obstruction groups vanish, and those which do not are reasonably computable.

Let us describe one application of this. These Lubin-Tate spectra E are even-periodic and Landweber exact, meaning they may be constructed from periodic complex cobordism MUP by means of a *genus* $\pi_0 MUP \to \pi_0 E$. It is then natural to ask to what extent this can be refined into a highly structured *orientation* $MUP \to E$. In particular, Ando [**And95**] introduced E-power operations to study this question; Ando-Hopkins-Strickland [**AHS04**] classified which coordinates on the formal group associated to E give rise to $\mathbb{H}_\infty$ orientations $MUP \to E$, i.e. orientations that respect E-power operations; Zhu [**Zhu20**] gave a general existence and uniqueness theorem for these coordinates; and Walker [**Wal09**], Möllers [**Möl11**], Hopkins-Lawson [**HL18**], and Hahn-Yuan [**HY20**] have studied the refinement to $\mathbb{E}_\infty$ orientations, producing existence results at height 1. By combining our machinery with some of the known structure of E-power operations, we are able to immediately deduce the following.

Theorem (Theorem 4.9). Let E be a Lubin-Tate spectrum of height $h \leq 2$. Then every $\mathbb{H}_\infty$ orientation $MUP \to E$ refines to an $\mathbb{E}_\infty$ orientation. □

This improves known results at height 1, and produces the first $\mathbb{E}_\infty$ periodic orientations at height 2.

1.2. Extended summary

A certain amount of technical work is necessary to develop everything in the generality we require, and the details may serve to obscure the general flow of ideas. In this section, we provide a general overview of the highlights of this thesis, omitting most of the technicalities. In addition, we give some extra examples in Subsection 1.2.3.

1.2.1. Conventions. Before continuing on, let us fix a few general categorical conventions.

We will freely use the theory of ∞-categories, which we refer to just as categories, as developed by Lurie in [**Lur17b**], and by default all of our constructions should be interpreted in this sense. We write $\mathcal{G}\mathrm{pd}_\infty$ for the category of ∞-groupoids, also commonly known as the (∞-)category of spaces, and for a small category $\mathcal{C}$ we write $\mathrm{Psh}(\mathcal{C})$ for the category of presheaves of ∞-groupoids on $\mathcal{C}$, writing instead $\mathrm{Psh}(\mathcal{C}, \mathcal{S}\mathrm{et})$ when we mean presheaves of sets, and similarly for presheaves valued in other categories.

We follow the standard convention of fixing a small universe of ∞-groupoids, with respect to which everything in sight will be at least locally small, contained in a universe of large ∞-groupoids, with respect to which everything in sight is small, unless otherwise specified. For a (locally small) category $\mathcal{C}$, we write $\mathrm{Psh}(\mathcal{C})$ for the category of presheaves on $\mathcal{C}$ that arise as small colimits of representable presheaves; this is the cocompletion of $\mathcal{C}$ under small colimits. We write $h\colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ for the Yoneda embedding, and write the same for various restricted Yoneda embeddings.

In general, given a functor $f\colon \mathcal{C} \to \mathcal{D}$, we write $f_!\colon \mathrm{Psh}(\mathcal{C}) \to \mathrm{Psh}(\mathcal{D})$ for the functor obtained from f by left Kan extension along the Yoneda embedding $h\colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$, and write the same for similar situations, such as $f_!\colon \mathrm{Psh}(\mathcal{C}) \to \mathcal{D}$ when $\mathcal{D}$ admits sufficiently many colimits.

For a category $\mathcal{C}$, we write $\mathrm{h}\mathcal{C}$ for the homotopy category of $\mathcal{C}$, and $\mathcal{C}^{\simeq}$ for the maximal sub-∞-groupoid of $\mathcal{C}$.

1.2.2. Homotopy. (Chapter 2).

Chapter 2 contains the main homotopical work of this thesis. In this chapter, we introduce the notion of a *loop theory*, and explain how these may be used to produce various obstruction theories and spectral sequences in homotopy theory.

1.2.2.1. *Algebraic theories.* (Section 2.1).

The basis of our work is a variant of the classic notion of a *Lawvere theory*, which is a categorical approach to universal algebra pioneered by Lawvere in this thesis [**Law04**]. Classically, a Lawvere theory may be defined as a category $\mathcal{C}$ with object set $\mathbb{N}$ wherein n is the n-fold coproduct of 1 for all $n \in \mathbb{N}$. The category of *models* of $\mathcal{C}$ is then the category of presheaves X on $\mathcal{C}$ such that the canonical map $X(n) \to X(1)^{\times n}$ is an isomorphism for all $n \in \mathbb{N}$.

By only asking that $\mathcal{C}$ has finite coproducts, and not that a specified object generated $\mathcal{C}$ under coproducts, we are led to the notion of a *(multisorted) algebraic theory*. Note that no particular sorts are specified in this definition; this approach emphasizes the aspects of algebraic theories which are invariant under Morita equivalence, i.e. emphasizes their categories of models as the primary objects of interest. Here, the category $\mathcal{M}\mathrm{odel}_{\mathcal{C}}$ of models of $\mathcal{C}$ is the category of presheaves X on $\mathcal{C}$ such that $X(\coprod_{i \in F} C_i) \simeq \prod_{i \in F} X(C_i)$ for any finite collection $\{C_i : i \in F\}$ of objects in $\mathcal{C}$.

By restricting $\mathcal{C}$ to be a *discrete* algebraic theory, that is, a 1-category, and considering only the full subcategory $\mathcal{M}\mathrm{odel}_{\mathcal{C}}^{\heartsuit} \subset \mathcal{M}\mathrm{odel}_{\mathcal{C}}$ of discrete models, that is, of set-valued models, one recovers from this a large number of naturally occuring algebraic categories. Taking $\mathcal{C}$ to still be a discrete algebraic theory, the category $\mathcal{M}\mathrm{odel}_{\mathcal{C}}$ of ∞-groupoid-valued models is a familiar homotopy theory: it is the underlying ∞-category of the category of simplicial set-valued models of $\mathcal{C}$ equipped with model structure constructed by Quillen [**Qui67**, Section II.4], as can be seen starting with work of Badzioch [**Bad02**], generalized by Bergner [**Ber06**], and put into the ∞-categorical context by Lurie [**Lur17b**, Section 5.5.9].

One can view the categories arising in this manner as exactly the categories of models of multisorted finite product theories, and this is useful for understanding various examples. For example, if $\mathcal{C}$ is the category of finitely generated and free abelian groups, then $\mathcal{C}$ is an algebraic theory, and the category of models of $\mathcal{C}$ is equivalent to the category of abelian groups; roughly, this is because an abelian group M is determined by its addition map, which may be recovered as the map $\Delta^* \colon \mathcal{A}\mathrm{b}(\mathbb{Z}, M) \times \mathcal{A}\mathrm{b}(\mathbb{Z}, M) \cong \mathcal{A}\mathrm{b}(\mathbb{Z} \oplus \mathbb{Z}, M) \to \mathcal{A}\mathrm{b}(\mathbb{Z}, M)$ given by restriction along the diagonal $\Delta \colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. For our purposes, it is more useful to view these categories as those which admit a family of compact projective generators; from this perspective, the category $\mathcal{M}\mathrm{odel}_{\mathcal{C}}$ is best characterized as the free cocompletion of $\mathcal{C}$ under filtered colimits and geometric realizations [**Lur17b**, Section 5.5.8].

We are interested in certain categories which admit families of projective, but not necessarily compact, generators. To incorporate these, we allow our theories to be *infinitary*. The classic reference for infinitary theories is Wraith [**Wra70**], although certain size issues are overlooked there. To deal with these, we restrict ourselves to *bounded* theories, i.e. those which are generated by κ-ary operations for some regular cardinal κ. This has the further benefit of making available to us all the tools from the theory of presentable categories.

In general, infinitary theories are not as well-behaved as finitary theories. In order to obtain a story mimicking the finitary case, we must restrict ourselves to those theories which are *Mal'cev*; see for instance [**Smi76**] and [**Lam92**], though we require very little of the general theory. This assumption turns out to play two roles: not only is it used to ensure that $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ has good properties when $\mathcal{P}$ is an infinitary theory, it is also necessary for applications of the further homotopical refinement of loop theories which we will introduce below.

Let us proceed to the precise definition. A *Mal'cev* operation on a set H is a ternary operation $t\colon H \times H \times H \to H$ satisfying $t(x, x, y) = y$ and $t(x, y, y) = x$; a set equipped with a Mal'cev operation is called a *herd*. The motivating example is $H = \mathrm{Iso}(X, Y)$ for two objects X and Y of some 1-category, with Mal'cev operation $t(f, g, h) = fg^{-1}h$; all of our examples are ultimately derived from this. Herds are modeled by a Lawvere theory, so it makes sense to speak of herd objects in arbitrary categories with finite products.

Definition (Definition 2.1). A *Mal'cev theory* is a category $\mathcal{P}$ such that

(1) $\mathcal{P}$ admits small coproducts;
(2) All objects of $\mathcal{P}$ admit the structure of a coherd.

The category of *models* of $\mathcal{P}$ is the category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ of presheaves X on $\mathcal{P}$ such that $X(\coprod_{i\in I} P_i) \simeq \prod_{i\in I} X(P_i)$ for any set $\{P_i : i \in I\}$ of objects in $\mathcal{P}$, and $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is the full subcategory of discrete, or set-valued, models. $\triangleleft$

This definition may be somewhat opaque. A similar notion was studied in [**Lur11b**, Section 4.2], only with groups in place of herds; from this perspective, herds arise as an unpointed generalization of groups. When $\mathcal{P}$ is a discrete theory, there is a much more elegant formulation: a discrete theory $\mathcal{P}$ is Mal'cev precisely when every simplicial set-valued model of $\mathcal{P}$ takes values in Kan complexes. This is exactly the condition Quillen requires in [**Qui67**, II.4] to produce homotopy theories of simplicial objects in non-compactly generated settings. We expect there could be more elegant or more general formulations of the Mal'cev condition for ∞-categorical theories. For this reason we gather the facts which rely on the Mal'cev assumption in one place in Subsection 2.1.1, after which it no longer appears explicitly, and everything we do holds equally well for any theory satisfying properties of the sort laid out there.

We will only be concerned with Mal'cev theories, and so will refer to them simply as *theories*. If $\mathcal{C}$ is a finitary theory and $\mathcal{P} \subset \mathcal{M}\mathrm{odel}_{\mathcal{C}}$ is generated by $\mathcal{C}$ under coproducts, then $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{C}}$ (Proposition 2.3); thus infinitary theories do indeed generalize finitary theories. Throughout the thesis, we will make some minor size assumptions, assuming that our theories are generated in a similar way by a small, but not necessary countable, amount of data (Remark 2.3).

Remark 1.1. All of the discrete theories we will encounter arise as a combination of the following facts:

(1) If $\mathcal{A}$ is a cocomplete abelian category and $\mathcal{P} \subset \mathcal{A}$ is a full subcategory consisting of projective objects and closed under coproducts such that every $M \in \mathcal{A}$ is resolved by objects of $\mathcal{P}$, then $\mathcal{A} \simeq \mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ (Proposition 3.2);
(2) If $\mathcal{P}$ is a discrete theory, T is a monad on $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ preserving reflexive coequalizers, and $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ is the full subcategory spanned by the image of $\mathcal{P}$ under T, then $T\mathcal{P}$ is a theory and $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Model}_{T\mathcal{P}}^{\heartsuit}$;
(3) If $\mathcal{P}$ is a discrete theory and $X \in \mathrm{Model}_{\mathcal{P}}^{\heartsuit}$, then the slice category $\mathcal{P}/X$ is a theory and $\mathrm{Model}_{\mathcal{P}/X}^{\heartsuit} \simeq \mathrm{Model}_{\mathcal{P}}^{\heartsuit}/X$. ◁

In Section 2.1, we show that (Mal'cev) theories and their categories of models indeed share all the good properties of finitary theories. Most importantly, in Subsection 2.1.1 we show that $\mathrm{Model}_{\mathcal{P}}$ is the free cocompletion of $\mathcal{P}$ under geometric realizations. Moreover, $\mathrm{Model}_{\mathcal{P}}$ is presentable under our mild size conditions on $\mathcal{P}$. In Subsection 2.1.2, we verify that if $\mathcal{P}$ is a discrete theory, then $\mathrm{Model}_{\mathcal{P}}$ is the underlying ∞-category of Quillen's model category of simplicial set-valued models of $\mathcal{P}$, and review the notion of left-derived functor available in this context.

1.2.2.2. *Loop theories.* (Section 2.2).

Many of the categories we care most about are not of the form $\mathrm{Model}_{\mathcal{P}}$. For example, no nontrivial stable category is of this form. Heuristically, theories are *product* theories, and so encode operations with arities indexed by *discrete* sets, whereas these categories require operations indexed over higher-dimensional objects, such as spheres. This leads to the following definition.

Definition (Definition 2.3). A theory $\mathcal{P}$ is a *loop theory* if for any finite wedge of spheres F and $P \in \mathcal{P}$, the tensor $F \otimes P = \operatorname{colim}_{x \in F} P$ exists in $\mathcal{P}$. If $\mathcal{P}$ is a loop theory, then $\mathrm{Model}_{\mathcal{P}}^{\Omega} \subset \mathrm{Model}_{\mathcal{P}}$ is the full subcategory of models X such that $X(F \otimes P) \simeq X(P)^F$ for all $P \in \mathcal{P}$ and finite wedge of spheres F. ◁

We can now describe the general philosophy of this chapter. For a great many categories $\mathcal{M}$ that arise in homotopy theory, one can find (possibly multiple) full subcategories $\mathcal{P} \subset \mathcal{M}$ which are loop theories such that $\mathcal{M} \simeq \mathrm{Model}_{\mathcal{P}}^{\Omega}$. Upon choosing such a $\mathcal{P}$, one may then embed $\mathcal{M}$ into the larger category $\mathrm{Model}_{\mathcal{P}}$, and this category provides a bridge between $\mathcal{M}$ and the essentially algebraic category $\mathrm{Model}_{\mathrm{h}\mathcal{P}}$. Conceptually, we may view $\mathrm{Model}_{\mathcal{P}}$ as a deformation with generic fiber $\mathcal{M}$ and special fiber $\mathrm{Model}_{\mathrm{h}\mathcal{P}}$; in particular, this category gives access to new filtrations on constructions in $\mathcal{M}$, with filtration quotients computed in $\mathrm{Model}_{\mathrm{h}\mathcal{P}}$. By studying these filtrations, one gains access to various obstruction theories and spectral sequences by which one may approach the homotopy theory of $\mathcal{M}$ starting with the algebra of $\mathrm{Model}_{\mathrm{h}\mathcal{P}}$.

In the finitary and pointed case, where the objects of $\mathcal{P}$ are instead asked to be homotopy cogroups, this context was first studied in general by Pstrągowski [**Pst17**], where it was used to give a conceptual approach to the realization problem for Π-algebras of Blanc-Dwyer-Goerss [**BDG04**]. The first instance we are aware of where a particular case of this context appears is in work of Hopkins-Lurie [**HL17**]. We add to this story by using a different class of theories, carrying out new constructions, and giving tools for recognizing additional examples. We view as one of the primary benefits of the framework the ease in which it is adapted to new situations, making the resulting computational tools more readily acessible.

In Section 2.2, we develop the basic properties of loop theories. Most important for the general theory is Pstrągowski's interpretation of the spiral sequence, which holds equally well in our setting (Theorem 2.3). This is the primary tool that allows us to identify various constructions in terms of the algebraic category $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$.

In Subsection 2.2.3, we give tools for constructing and identifying examples. Heuristically, $\mathcal{M} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ whenever $\mathcal{P} \subset \mathcal{M}$ is a reasonable collection of free objects closed under coproducts and S^1-tensors. Here, "free" does not have a precise meaning; $\mathcal{P}$ must be a loop theory, but otherwise there is no particular freeness condition imposed on its objects. For example, if $\mathcal{M}$ is a compactly generated stable category, then $\mathcal{M} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ for any full subcategory $\mathcal{P} \subset \mathcal{M}$ which contains a family of compact generators and is a loop theory closed under coproducts, suspensions, and desuspensions in $\mathcal{M}$ (Theorem 2.4). In particular, there can be very different choices of $\mathcal{P} \subset \mathcal{M}$, producing wildly different algebraic categories $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$, for which one has $\mathcal{M} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$; we give an example below in Subsubsection 1.2.3.5.

Remark 1.2. Before continuing on, we pause to point out the following notational subtlety. By way of example, let R be an $\mathbb{A}_\infty$-ring, and let $\mathcal{P} = \mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{free}}$. In this case it turns out

$$\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_R, \qquad \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{R_*}^{\heartsuit},$$

and the following diagram commutes:

$$\begin{tikzcd} \mathrm{L}\mathcal{M}\mathrm{od}_R \arrow[r, "\pi_*"] \arrow[d, "h"] & \mathrm{L}\mathcal{M}\mathrm{od}_{R_*}^{\heartsuit} \arrow[d, "h", "\simeq"'] \\ \mathcal{M}\mathrm{odel}_{\mathcal{P}} \arrow[r, "\pi_0"] & \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \end{tikzcd}.$$

The notational subtlety is that

$$\pi_* M = \pi_0 h(M).$$

In general, if $\mathcal{P}$ is a loop theory and $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, then $\pi_0 X$ encodes all of the homotopy groups of X. ◁

1.2.2.3. *Stable loop theories.* (Section 2.3).

Section 2.3 and Section 2.4 are independent of each other, and both use the categories $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ to produce computational tools for $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$. We begin by describing the contents of Section 2.3.

Definition. A loop theory $\mathcal{P}$ is said to be *stable* if

(1) $\mathcal{P}$ is pointed and admits suspensions;

(2) $\Sigma\colon \mathcal{P} \to \mathcal{P}$ is an equivalence. ◁

Fix a stable loop theory $\mathcal{P}$. Then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is additive and thus embeds into its stabilization, which can be identified as the category $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ of $\mathcal{S}\mathrm{p}$-valued models of $\mathcal{P}$. In this context, we write $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}}$ for the category of $\mathcal{S}\mathrm{p}_{\geq 0}$-valued models of $\mathcal{P}$ and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ for the category of abelian group-valued models of $\mathcal{P}$; the forgetful functors then allow us to identify $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$.

There is in addition a category $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$ of $\mathcal{S}\mathrm{p}$-valued models X such that $X(F \otimes P) \simeq X(P)^F$ for $P \in \mathcal{P}$ and F a finite wedge of spheres; it is equivalent to ask just that $X(\Sigma P) \simeq \Omega X(P)$ for $P \in \mathcal{P}$. Moreover, the functor $\Omega^\infty\colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ restricts to an equivalence of categories $\Omega^\infty\colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$. However, there is an important subtlety in this: $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ and $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ are very different subcategories of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$.

For $X, Y \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, there is a mapping spectrum $\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X, Y)$, so that $\Omega^\infty \mathcal{E}\mathrm{xt}_{\mathcal{P}}(X, Y) = \mathrm{Map}_{\mathcal{P}}(X, Y)$. Despite the above subtlety, if $N \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$, $M \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, and LM is the preimage of M under the equivalence $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, then $\mathcal{E}\mathrm{xt}_{\mathcal{P}}(M, N) \simeq \mathcal{E}\mathrm{xt}_{\mathcal{P}}(LM, N)$. In Subsection 2.3.4, we describe the decomposition of these mapping spectra that may be obtained from Postnikov towers in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$. This takes the form of the following universal coefficient spectral sequence.

For $X \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{P}}$ and $n \in \mathbb{Z}$, write $X[n]$ for the model obtained by restricting X along the functor $\mathrm{h}\mathcal{P} \to \mathrm{h}\mathcal{P}$ induced by $\Sigma^n\colon \mathcal{P} \to \mathcal{P}$, so $\pi_n X \cong (\pi_0 X)[n]$ if $X \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$.

Theorem (Theorem 2.9). Fix $M \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ and $N \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$. Then the spectral sequence associated to the Postnikov decomposition

$$\mathcal{E}\mathrm{xt}_{\mathcal{P}}(M, N) \simeq \lim_{n \to \infty} \mathcal{E}\mathrm{xt}_{\mathcal{P}}(M, \tau_{\leq n} N)$$

is of signature

$$E_1^{p,q} = \mathrm{Ext}_{\mathrm{h}\mathcal{P}}^{p+q}(\pi_0 M; \pi_0 N[p]) \Rightarrow \pi_{-q} \mathcal{E}\mathrm{xt}_{\mathcal{P}}(M, N),$$

with differential $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q+1}$. ◁

For an arbitrary loop theory $\mathcal{P}$, the category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is a localization of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$; the distinguishing feature of the stable case is that the localization $L\colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$ admits an explicit description (Theorem 2.6). Given $X \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, write X_{Σ^n} for the model defined by $X_{\Sigma^n}(P) = X(\Sigma^n P)$. Then $L\colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$ may be identified as

$$LX = \operatorname*{colim}_{n\to\infty} \Sigma^{-n} X_{\Sigma^{-n}}.$$

We apply this in Subsection 2.3.2; to describe this application we require some additional notation.

Fix another loop theory $\mathcal{P}'$, which need not be stable, together with some functor $F\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\Omega} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ which preserves geometric realizations. Then F is determined by its restriction f to $\mathcal{P}'$; explicitly, where $f_!\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is obtained from f by left Kan extension, there is an equivalence $Lf_!X \simeq FX$ for $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\Omega}$. The composite $\pi_0 \circ f\colon \mathcal{P}' \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ uniquely factors through the homotopy category to give $\overline{f}\colon \mathrm{h}\mathcal{P}' \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, and by left Kan extension this gives functor $\overline{F}\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ preserving reflexive coequalizers, and this admits a total left-derived functor $\mathbb{L}\overline{F} = \overline{f}_!\colon \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$ (cf. Proposition 2.4).

Theorem (Theorem 2.7). In the situation of the previous paragraph, given $R \in \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\Omega}$, the spectral sequence in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ associated to the tower

$$FR \simeq \operatorname*{colim}_{n\to\infty} \Sigma^{-n}(f_!R)_{\Sigma^{-n}}$$

is of signature

$$E^1_{p,q} = (\mathbb{L}_{p+q}\overline{F}\pi_0 R)[-p] \Rightarrow (\pi_0 FR)[q], \qquad d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q-1}.$$

This spectral sequence converges, for instance, if π_0 preserves filtered colimits or if each $\mathbb{L}\overline{F}\pi_0 R$ is truncated. $\triangleleft$

As described, the localization $L\colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$ naturally factors through a functor $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{Fun}(\mathbb{Z}, \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}})$. There is natural notion of a *monoidal loop theory* (Definition 2.4), and this functor turns out to be lax monoidal whenever $\mathcal{P}$ is monoidal. This leads to the introduction of pairings on spectral sequences constructed in the above manner (Theorem 2.8). For example, this gives rise to pairings on Künneth-type spectral sequences; such pairings have a history of being difficult to construct by hand [**Til16**].

1.2.2.4. *Postnikov decompositions.* (Section 2.4).

In Section 2.4, we consider Postnikov decompositions in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$, where $\mathcal{P}$ is a loop theory which need not be stable. The content of this section is more closely related to the earlier simplicial work of Blanc-Dwyer-Goerss [**BDG04**] and Goerss-Hopkins [**GH05**], and directly builds on Pstrągowski [**Pst17**]. One consequence of the theory is the following.

Theorem (Theorem 2.11). Fix $A, C \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, together with a map $\phi\colon \pi_0 A \to \pi_0 C$ in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. Let $\mathrm{Map}_{\mathcal{P}}^{\phi}(A, C) \subset \mathrm{Map}_{\mathcal{P}}(A, C)$ be the space of maps f such that $\pi_0 f = \phi$. Then the Postnikov tower of C in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ gives a decomposition

$$\mathrm{Map}_{\mathcal{P}}^{\phi}(A, C) \simeq \lim_{n\to\infty} \mathrm{Map}_{\mathcal{P}}^{\phi}(A, C_{\leq n}),$$

where $\mathrm{Map}_{\mathcal{P}}^{\phi,\leq 0}(A, C) = \{\phi\}$ and for each $n \geq 1$ there is a canonical fiber sequence

$$\mathrm{Map}_{\mathcal{P}}^{\phi}(A, C_{\leq n}) \to \mathrm{Map}_{\mathcal{P}}^{\phi}(A, C_{\leq n-1}) \to \mathrm{Map}_{\mathrm{h}\mathcal{P}/\pi_0 C}(\pi_0 A; B_{\pi_0 C}^{n+1}\Pi_n C),$$

where we have written $\Pi_n C = \pi_0 C^{S^n} \in \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}/\pi_0 C)$ and $B_{\pi_0 C}^{n+1}\Pi_n C$ for its delooping in the slice category $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}/\pi_0 C$. In particular, there are successively defined obstructions in the Quillen cohomology groups $H_{\mathrm{h}\mathcal{P}/\pi_0 C}^{n+1}(\pi_0 A; \Pi_n C) = \pi_0 \mathrm{Map}_{\mathrm{h}\mathcal{P}/\pi_0 C}(\pi_0 A; B_{\pi_0 C}^{n+1}\Pi_n C)$ to realizing ϕ as arising from a map $A \to C$. ◁

In fact we prove something stronger, giving the analogous decomposition for mapping spaces in slice categories of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

In Subsection 2.4.4, we verify that the obstruction theory of [**Pst17**] for realizing an object $\Lambda \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ as $\Lambda = \pi_0 R$ with $R \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ holds in our setting (Theorem 2.12). The short form of this is that there are successively defined obstructions in certain quotients of $H_{\mathrm{h}\mathcal{P}/\Lambda}^{n+2}(\Lambda; \Lambda\langle n\rangle)$, where $\Lambda\langle n\rangle(P) = \Lambda(S^n \otimes P)$, to producing such an R (Proposition 2.16).

1.2.2.5. *Miscellaneous.* (Section 2.5, Section 2.6).

Section 2.5 discusses localizations and completions in the context of theories.In Subsection 2.5.1, we describe some general facts about localizations of theories. In Subsection 2.5.2, we introduce R-linear theories for a connective $\mathbb{E}_2$-ring R, and study the corresponding notion of I-completions for a finitely generated ideal $I \subset R_0$. In particular, we give conditions under which the algebraic categories arising in this context may be described more explicitly. The examples which may be obtained in this context were the original motivation for working with infinitary theories.

Section 2.6 can be considered an appendix to Section 2.3. In the latter, we required some properties of the spectral sequence associated to a tower in a stable category with t-structure; in particular, we required the relation between pairings of towers and pairings of their spectral sequences. We review the construction and convergence of these spectral sequences in Subsection 2.6.1, and give an overview of their multiplicative properties in Subsection 2.6.2 and Subsection 2.6.3.

1.2.3. Examples. The presentation of Chapter 2 is, by necessity, somewhat abstract. The following are some examples that fit into the framework developed (cf. Subsection 2.2.3).

1.2.3.1. *Modules over ring spectra.* Let R be an $\mathbb{A}_\infty$ ring spectrum, and let $\mathcal{R} = \mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{free}}$ be the full subcategory of $\mathrm{L}\mathcal{M}\mathrm{od}_R$ generated by R under small coproducts, suspensions, and desuspensions. Then $\mathcal{R}$ is a loop theory, and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{R}}^{\Omega} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_R$; moreover $\mathrm{h}\mathcal{R} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{R_*}^{\mathrm{free}}$, and therefore $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{R}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{R_*}$ is the unbounded derived category of left R_*-modules.

The category $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{R}}$ of all models is more exotic. This category fits into a span

$$\mathrm{L}\mathcal{M}\mathrm{od}_{R_*} \xleftarrow{\pi_!} \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{R}} \xrightarrow{L} \mathrm{L}\mathcal{M}\mathrm{od}_R \ ,$$

and we have proposed to consider it a deformation with generic fiber $\mathrm{L}\mathcal{M}\mathrm{od}_R$ and special fiber $\mathrm{L}\mathcal{M}\mathrm{od}_{R_*}$. There is another method for building deformations of stable homotopy theories that is perhaps more concrete, which proceeds via *filtered objects*; see for instance [**GIKR18**] for an application of this philosophy. The category $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{R}}$ turns out to admit a description in terms of filtered spectra: there is an equivalence $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{R}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{W(R)}$, where $W(R)$ is the Whitehead tower of R, viewed as a filtered ring spectrum. We expect that various other categories of the form $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ admit similar descriptions using filtered objects.

Remark 1.3. Let X be a filtered spectrum, and set $X(\infty) = \operatorname{colim}_p X(p)$. Then there is filtration spectral sequence of signature

$$E^1_{p,q} = \pi_q \operatorname{Cof}(X(p-1) \to X(p)) \Rightarrow \pi_q X(\infty), \qquad d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q-1}.$$

Examples from the filtered approach to deformations suggest that to understand $\pi_* X(\infty)$ as computed via this filtration spectral sequence, one should go further and consider all of the intermediate objects $X(p)$. At a basic level, this amounts to studying the bigraded homotopy groups $\pi_{s,c}X = \pi_s X(c)$. Each $\pi_{s,*}X$ is a module over a graded polynomial ring $\mathbb{Z}[\sigma]$, where $\sigma\colon \pi_{s,c-1}X \to \pi_{s,c}X$ is induced by $X(c-1) \to X(c)$; this satisfies $\pi_{s,*}X[\sigma^{-1}] = \pi_s X(\infty) \otimes \mathbb{Z}[\sigma^{\pm 1}]$, and in general the $\mathbb{Z}[\sigma]$-module structure of $\pi_{*,*}X$ tracks information about the computation of $\pi_* X(\infty)$ via the filtration spectral sequence, such as differentials and hidden extensions. All of the obstruction theories and spectral sequences we construct, both stably and unstably, proceed by constructing some filtered or cofiltered object with identifiable filtration quotients, and so we may directly import these insights into our setting. For example, in the context of Theorem 2.7, one might instead compute $\pi_*(f_! R)$ itself. $\triangleleft$

1.2.3.2. *Modules over equivariant ring spectra.* Let G be a finite group, $\mathcal{S}\mathrm{p}^G$ be the category of genuine G-equivariant spectra, and R be an $\mathbb{A}_\infty$ ring in $\mathcal{S}\mathrm{p}^G$. Let $\mathcal{R} \subset \mathrm{L}\mathcal{M}\mathrm{od}_R$ be the full subcategory generated under coproducts by objects of the form $\Sigma^\alpha R \otimes S_+$ for $\alpha \in RO(G)$ and S a finite G-set. Then $\mathcal{R}$ is a loop theory, $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{R}}^{\Omega} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_R$, and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{R}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{R_\star}$ is the unbounded derived category of left Mackey modules over the $RO(G)$-graded Green functor $R_\star$. In fact we would still have $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{R}}^{\Omega} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_R$ had we restricted α to $\mathbb{Z} \subset RO(G)$, only in this case $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{R}}^{\heartsuit}$ would be the category of left Mackey modules over the $\mathbb{Z}$-graded Green functor R_*.

Now suppose for simplicity that R is an $\mathbb{E}_2$ ring, so that $\mathrm{L}\mathcal{M}\mathrm{od}_R$ is a monoidal category. The monoidal product on $\mathrm{L}\mathcal{M}\mathrm{od}_R$ restricts to $\mathcal{R}$, which then extends to a monoidal product on $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{R}}$. In addition the monoidal product on $\mathcal{R}$ induces one on $\mathrm{h}\mathcal{R}$, which then extends to a monoidal product on $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{R}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{R_\star}$, which is simply the (derived) box product over $R_\star$. Now fix $M, N \in \mathrm{L}\mathcal{M}\mathrm{od}_R$, giving models $h(M), h(N) \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}}$. The methods of Theorem 2.7 applied to $h(M) \otimes h(N)$ then give a Mackey functor Künneth spectral sequence

$$E^1_{p,q} = \pi_{p+q}(\pi_\star M \square^{\mathbb{L}}_{\pi_\star R} \pi_\star N)_{\star - p} \Rightarrow \pi_{\star+q}(M \otimes_R N), \qquad d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q-1},$$

such as those considered in [**LM06**]. Moreover, one has access to all the pairings on these spectral sequences that one could hope for.

As stated, Theorem 2.9 gives a universal coefficient spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^{p+q}_{R_\star}(\pi_\star M; \pi_{\star+p} N) \Rightarrow \pi_{-q} \mathcal{E}\mathrm{xt}_R(M,N), \qquad d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q+1}$$

of abelian groups. By naturality, as discussed at the end of Subsection 2.3.4, this may be enhanced to a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^{p+q}_{R_\star}(\pi_\star M; \pi_{\star+p} N)_\star \Rightarrow \pi_{\star-q} \mathcal{E}\mathrm{xt}_R(M,N), \qquad d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q+1}$$

internal to the category of $R_\star$-modules.

1.2.3.3. *Functor categories.* Let $\mathcal{P}$ be a loop theory and $\mathcal{J}$ be a 1-category. Then

$$\mathrm{Fun}(\mathcal{J}, \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}) \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}^{\mathcal{J}}}^{\Omega}, \qquad \mathrm{Fun}(\mathcal{J}, \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}) \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}^{\mathcal{J}}}^{\heartsuit},$$

where $\mathcal{P}^{\mathcal{J}}$ is the loop theory obtained as the image of the representable functors under the composite

$$\mathrm{Fun}(\mathcal{J} \times \mathcal{P}^{\mathrm{op}}, \mathcal{G}\mathrm{pd}_\infty) \simeq \mathrm{Fun}(\mathcal{J}, \mathrm{Psh}(\mathcal{P})) \to \mathrm{Fun}(\mathcal{J}, \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}),$$

where the second functor is obtained from localization map $\mathrm{Psh}(\mathcal{P}) \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$. Explicitly, for $P \in \mathcal{P}$ and $i \in \mathcal{J}$, define $H_{P,i} \colon \mathcal{J} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ by

$$H_{P,i}(j) = \mathcal{J}(i,j) \otimes h(P) = h\left(\coprod_{x \in \mathcal{J}(i,j)} P\right);$$

then $\mathcal{P}^{\mathcal{J}}$ is generated under coproducts in the category $\mathrm{Fun}(\mathcal{J}, \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega})$ by functors of the form $H_{P,i}$. The assumption that $\mathcal{J}$ is a 1-category is necessary to identify $\mathrm{h}(\mathcal{P}^{\mathcal{J}}) \simeq (\mathrm{h}\mathcal{P})^{\mathcal{J}}$.

Write $q^*\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \to \mathrm{Fun}(\mathcal{J}, \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega})$ for the diagonal. Given $F\colon \mathcal{J} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ and $P \in \mathcal{P}$, there is an equivalence

$$\left(\lim_{j\in\mathcal{J}} F(j)\right)(P) \simeq \mathrm{Map}_{\mathcal{P}}(q^*h(P), F).$$

Applying Theorem 2.11, or Theorem 2.9 if $\mathcal{P}$ is stable, in this context then returns a homotopy limit spectral sequence for computing $\pi_*\left(\lim_{j\in\mathcal{J}} F(j)\right)$. If $\mathcal{P}$ is stable, then applying Theorem 2.7 to the colimit functor $\mathrm{Fun}(\mathcal{J}, \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}) \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ returns a homotopy colimit spectral sequence.

For example, when $\mathcal{J} = BG$ for a discrete group G, these return homotopy fixed point and homotopy orbit spectral sequences respectively. When $\mathcal{J} = \Delta^{\mathrm{op}}$, the homotopy colimit spectral sequence recovers the standard spectral sequence of a simplicial object. The fact that geometric realizations respect monoidal structures then combines with Theorem 2.8 to produce the standard pairings on these spectral sequences.

1.2.3.4. *Completed modules.* The following example is developed in further detail in Subsection 2.5.2. Let R be an $\mathbb{E}_2$ ring and $I \subset \pi_0 R$ be a finitely generated ideal. Then there is a category $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$ of I-complete R-modules, and a category of $\mathrm{L}\mathcal{M}\mathrm{od}_{R_*}^{\mathrm{Cpl}(I)}$ of "derived I-complete" R_*-modules. For example, taking $R = E$ to be a Lubin-Tate spectrum of height h and $I = \mathfrak{m} \subset E_0$ to be the maximal ideal, the category $\mathcal{M}\mathrm{od}_E^{\mathrm{Cpl}(\mathfrak{m})}$ recovers the category $\mathcal{M}\mathrm{od}_E^{\mathrm{loc}}$ of $K(h)$-local E-modules, and $\mathcal{M}\mathrm{od}_{E_*}^{\mathrm{Cpl}(\mathfrak{m})}$ is the derived category of L-complete E_*-modules in the sense of [**HS99**, Appendix A].

Let $\mathcal{P} = \mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I),\mathrm{free}}$ be the category of I-completions of free R-modules. Then $\mathcal{P}$ is a loop theory, and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$. By contrast, one subtlety of completions is that it is not always the case that $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{P}} \simeq \mathcal{M}\mathrm{od}_{R_*}^{\mathrm{Cpl}(I)}$. This does however hold under a minor algebraic tameness condition on $I \subset R_*$ which is satisfied in practice, at least in situations where one would wish to compute in $\mathcal{M}\mathrm{od}_{R_*}^{\mathrm{Cpl}(I)}$.

1.2.3.5. *Bocksteins.* If $\mathcal{P}$ is a theory, then $\mathcal{P} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is in effect a distinguished subcategory: the idempotent completion of $\mathcal{P}$ is equivalent to the full subcategory of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ consisting of those models X which are projective in the sense that $\mathrm{Map}_{\mathcal{P}}(X, -)$ preserves geometric realizations (Proposition 2.2). By contrast, if $\mathcal{P}$ is a loop theory, then $\mathcal{P} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ carries no particular universal property, and wildly different loop theories can model the same category. Here is an example.

For simplicity, let R be an $\mathbb{E}_\infty$ ring such that $\pi_* R$ is Noetherian, and let $I \subset R_0$ be an ideal generated by a sequence $(u_0, \ldots, u_{h-1})$ which is regular on R_*, and with respect to which R is complete. By the preceding example, there is a loop theory $\mathcal{P} = \mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I),\mathrm{free}}$

with $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \simeq \mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$. The relevant algebraic tameness conditions are satisfied to ensure $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{P}} \simeq \mathcal{M}\mathrm{od}_{R_*}^{\mathrm{Cpl}(I)}$.

On the other hand, define

$$C(I) = C(u_0) \otimes_R \cdots \otimes_R C(u_{h-1}), \qquad C(u) = \mathrm{Cof}(u\colon R \to R).$$

Then $C(I)$ turns out to be a compact generator of $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$, and so there is an equivalence

$$\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)} \simeq \mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{E}\mathrm{xt}_R(C(I),C(I))}, \qquad M \mapsto \mathcal{E}\mathrm{xt}_R(C(I), M).$$

Here, we should take the product on $\mathcal{E}\mathrm{xt}_R(C(I), C(I))$ which is opposite to the standard composition product, or else consider right modules instead. In the language of loop theories, this equivalence may be understood as follows. Let $\mathcal{P}' \subset \mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$ be the full subcategory generated by $C(I)$ under small coproducts, suspensions, and desuspensions. Because $C(I)$ is a compact generator of $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$, there is an equivalence $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)} \simeq \mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{P}'}^{\Omega}$. On the other hand, $\mathcal{P}'$ may be identified as the theory associated to $\mathcal{E}\mathrm{xt}_R(C(I), C(I))$ via the construction of Subsubsection 1.2.3.1, and therefore $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{P}'}^{\Omega} \simeq \mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{E}\mathrm{xt}_R(C(I),C(I))}$.

The algebraic category we extract from this choice of loop theory is quite different from $\mathcal{M}\mathrm{od}_{R_*}^{\mathrm{Cpl}(I)}$. Let $k_* = (\pi_* R)/I$. Regularity of I implies that $k_* \cong \pi_* C(I)$, and that

$$\pi_*\mathcal{E}\mathrm{xt}_R(C(I), C(I)) \cong \Lambda_{k_*}(Q_0, \ldots, Q_{h-1}), \qquad |Q_i| = -1,$$

where Λ indicates an exterior algebra; here Q_i is a *Bockstein* element associated to u_i. Thus $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{P}'} \simeq \mathcal{L}\mathcal{M}\mathrm{od}_{\Lambda_{k_*}(Q_0,\ldots,Q_{h-1})}$.

Thus, whereas $\mathcal{P}$ leads one to approximate $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$ via complete R_*-modules, $\mathcal{P}'$ leads one to approximate $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$ via k_*-modules equipped with Bockstein information. To illustrate this, first note that as $\mathcal{E}\mathrm{xt}_R(C(I), M) \simeq \Sigma^{-h} M \otimes C(I)$, we may consider the equivalence $\mathcal{M}\mathrm{od}_R \simeq \mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{E}\mathrm{xt}_R(C(I),C(I))}$ as instead realized by $M \mapsto M \otimes C(I)$. Now if $M \in \mathcal{L}\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$, then the universal coefficient spectral sequence of Theorem 2.9 applied to $M \simeq \mathcal{E}\mathrm{xt}_R(R, M)$ using the theory $\mathcal{P}'$ takes the form

$$E_1^{p,q} = \mathrm{Ext}^{p+q}_{\Lambda_{k_*}(Q_0,\ldots,Q_{h-1})}(k_*, \pi_{*+p}(M \otimes C(I))) \Rightarrow \pi_{*-q} M.$$

By taking a Koszul resolution of k_*, we may step back a page and view this as being of the form

$$\pi_*(M \otimes C(I))[u_0, \ldots, u_{h-1}] \Rightarrow \pi_* M,$$

where each u_i lies in stem 0 and filtration 1. This is nothing more than the Bockstein spectral sequence for $\pi_* M$ with respect to $(u_0, \ldots, u_{h-1})$.

Neither $\mathcal{P}$ nor $\mathcal{P}'$ is more useful than the other; instead, they both shed light on different aspects of $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(I)}$. For example, we will use the first form in our applications to Lubin-Tate spectra in Section 4.2, whereas Hopkins-Lurie [**HL17**] makes use of (a choice-free variant of) the second form, also with applications to Lubin-Tate spectra.

1.2.3.6. *Algebras over monads.* Let $\mathcal{P}$ be a loop theory, and let T be a monad on $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ which preserves geometric realizations. Define $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ to be the full subcategory spanned by the essential image of $\mathcal{P}$ under T. Then $T\mathcal{P}$ is a loop theory, and there is an equivalence $\mathcal{A}\mathrm{lg}_T \simeq \mathcal{M}\mathrm{odel}_{T\mathcal{P}}^{\Omega}$. The condition that T preserves geometric realizations may be relaxed at the expense of a more technical statement; this is all treated in Subsection 2.2.3. The theory $\mathrm{h}T\mathcal{P}$ should be thought of as the theory of *homotopy operations* for T-algebras (cf. Proposition 3.1); compare the treatments of theories of power operations in [**Rez06**] and [**Law19**].

This gives rise to more examples than we could hope to enumerate. Two examples in particularly good standing are $\mathcal{P} = \mathcal{M}\mathrm{od}_{H\mathbb{F}_p}^{\mathrm{free}}$ and T the free $\mathbb{E}_\infty$ algebra functor, and $\mathcal{P} = \mathcal{M}\mathrm{od}_E^{\mathrm{loc,free}}$ for E a Lubin-Tate spectrum of height h and T the free $K(h)$-local $\mathbb{E}_\infty$ algebra functor. These are the subject of Chapter 4.

1.2.3.7. *Associative algebras.* Let R be an $\mathbb{E}_2$ ring, so that $\mathrm{L}\mathcal{M}\mathrm{od}_R$ is monoidal and there is a category $\mathcal{A}\mathrm{lg}_R = \mathcal{M}\mathrm{on}(\mathrm{L}\mathcal{M}\mathrm{od}_R)$ of $\mathbb{A}_\infty$ algebras over R. Let $\mathcal{P} = \mathcal{A}\mathrm{lg}_R^{\mathrm{free}}$; then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \simeq \mathcal{A}\mathrm{lg}_R$ and $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit} \simeq \mathcal{A}\mathrm{lg}_{R_*}^{\heartsuit}$. In this context Theorem 2.11 is an obstruction theory for mapping spaces in $\mathcal{A}\mathrm{lg}_R$ built from the ordinary Hochschild cohomology of R_*-algebras. By taking R to instead be a G-equivariant $\mathbb{E}_2$ ring, we would obtain an obstruction theory built from the cohomology of associative Green algebras for the commutative Green functor $R_\star$.

1.2.3.8. *$\mathbb{F}_p$-synthetic p-profinite spaces.* We end with an example of a different flavor.

Fix a prime p, and let $\mathcal{P}_0^{\mathrm{op}} \subset \mathcal{G}\mathrm{pd}_\infty$ be the full subcategory spanned by finite products of the Eilenberg-MacLane spaces $K(\mathbb{F}_p, n)$. Then $\mathcal{P}_0$ is a finitary loop theory; we may complete it to a full loop theory $\mathcal{P}$ by declaring $\mathcal{P} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}_0}$ to be the full subcategory generated by $\mathcal{P}_0$ under coproducts. Now consider $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\mathrm{op}}$; we propose to consider this a category of $\mathbb{F}_p$-synthetic p-profinite spaces. This is in analogy with the category of $\mathbb{F}_p$-synthetic spectra in the sense of Pstrągowski [**Pst18**], which is a deformation of the category of spectra refining the classic mod p Adam spectral sequence; see in particular [**BHS19**, Appendix A].

Let $\mathcal{F}\mathrm{in}_p \subset \mathcal{G}\mathrm{pd}_\infty$ be the full subcategory of p-finite spaces, i.e. those X such that $\pi_0 X$ is finite, X is truncated, and each $\pi_n(X,x)$ is a finite p-group. There is then an equivalence of categories $\mathrm{Pro}(\mathcal{F}\mathrm{in}_p) \simeq (\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega})^{\mathrm{op}}$, and $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$ is equivalent to the category of unstable algebras over the Steenrod algebra (cf. Subsection 4.1.3 and Example 4.3). In this context Theorem 2.11 may be interpreted as an unstable Adams spectral sequence.

Now let $\mathcal{P}'$ be defined in the same manner as $\mathcal{P}$, only using pointed objects throughout. To any pointed space X, we may define a model $h(X)$ of $\mathcal{P}'$ by $h(X)(P) = \mathrm{Map}(X, P)$, and

$X_p^\wedge = \mathrm{Map}_{\mathcal{P}'}(h(X), h(S^0))$ recovers the p-profinite completion of X. Define

$$\pi_{s,c}^{\mathbb{F}_p} X = \pi_s \operatorname{Fib}(X_p^\wedge \to \mathrm{Map}_{\mathcal{P}'}(h(X), h(S^0)_{\leq c-1})).$$

Although we are now working unstably, the ideas mentioned in Remark 1.3 still indicate that $\pi_{*,*}^{\mathbb{F}_p} X$ may be viewed as a deformation of $\pi_* X$ associated to the unstable Adams spectral sequence. Thus we may consider $\pi_{*,*}^{\mathbb{F}_p} X$ as a candidate definition for the $\mathbb{F}_p$-synthetic *unstable* homotopy groups of X.

1.2.4. Algebra. (Chapter 3).

The obstruction groups that appear in Chapter 2 are given in terms of formally defined algebraic theories of operations. In some cases, these theories describe familiar algebraic structures, such as modules over a ring; in other cases, these theories are sufficiently complicated that computations are entirely inaccessible. There is a third case, where these theories describe exotic algebraic structures which are nonetheless sufficiently well-behaved that the corresponding obstruction groups may be more explicitly understood. Two important examples of this are the theories of power operations for $\mathbb{E}_\infty$ algebras in positive characteristic, and for $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum of height h. These two examples turn out to share a number of formal similarities, and the purpose of Chapter 3 is to describe the general algebraic context allowing one to compute with examples such as these.

This chapter is essentially algebraic in nature, being concerned almost entirely with discrete theories. Some of this algebra is interesting in its own right, and so we have written this chapter in such a way that it may be read independently of Chapter 2.

In all the following, $\mathcal{P}$ will refer to a discrete theory.

1.2.4.1. *Algebras over theories and other topics.* (Section 3.1).

There are two fundamental facts which serve to make the theory of power operations for $\mathbb{E}_\infty$ algebras in positive characteristic, or for $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum of height h, well-behaved, from the perspective of the obstruction groups we wish to compute. By way of example, let DL be the theory of power operations for $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$; abstractly, this theory may be defined as $\mathrm{DL} = \mathrm{h}\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$. In addition, write $\mathcal{R}\mathrm{ing}_{\mathrm{DL}} = \mathcal{M}\mathrm{odel}_{\mathrm{DL}}$. The two key properties of this theory are the following:

(1) There is a forgetful functor $\mathcal{R}\mathrm{ing}_{\mathrm{DL}}^\heartsuit \to \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathbb{F}_{p*}}^\heartsuit$ which is both monadic and comonadic;
(2) The free models of DL have discrete and projective Quillen homology as $\mathbb{F}_{p*}$-rings.

The first fact implies that $\mathcal{R}\mathrm{ing}_{\mathrm{DL}}^\heartsuit$ is built in a simple way from the more familiar category $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathbb{F}_{p*}}^\heartsuit$ of graded commutative $\mathbb{F}_p$-rings, and the second fact implies that this continues to hold at the level of derived categories. Using these one may reduce the Quillen cohomology of DL-rings into a purely classical part, determined by the ordinary André-Quillen homology of

$\mathbb{F}_{p*}$-rings, and a purely linear part, governed by a suitable algebra of Dyer-Lashof operations. The use of commutative rings in particular is not necessary here, and instead we are led to the more general notion of an *algebra over a theory*, due to Freyd [**Fre66**] and Wraith [**Wra71**].

Definition (Definition 3.2). Let $\mathcal{P}$ be a theory. A (discrete) $\mathcal{P}$-*algebra* F consists of either of the following equivalent pieces of data:

(1) A colimit-preserving monad F on $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$;
(2) A limit-preserving $F^{\vee}$ on $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$.

In this context, there is an adjunction $F \dashv F^{\vee}$, and $\mathcal{A}\mathrm{lg}_F \simeq \mathrm{Co}\mathcal{A}\mathrm{lg}_{F^{\vee}}$. $\triangleleft$

This definition may be understood as follows. Let F be a monad on $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$, and let $\mathrm{Model}_F^{\heartsuit}$ be the category of models for the theory $F\mathcal{P} \subset \mathcal{A}\mathrm{lg}_F$. Suppose moreover that F preserves reflexive coequalizers; this is necessary to ensure $\mathrm{Model}_F^{\heartsuit} \simeq \mathcal{A}\mathrm{lg}_F$. Given $P, P' \in \mathcal{P}$, the sets $F(P)(P')$ describe *natural operations* on F-models. Explicitly, where ev_P denotes evaluation at P, there is a natural isomorphism

$$F(P)(P') \cong \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Model}_F^{\heartsuit}, \mathcal{S}\mathrm{et})}(\mathrm{ev}_P, \mathrm{ev}_{P'});$$

this is a consequence of the Yoneda lemma (cf. Proposition 3.1).

As F preserves reflexive coequalizers, for F to be an algebra it is sufficient that F preserves coproducts. As

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{Model}_F^{\heartsuit}, \mathcal{S}\mathrm{et})}\left(\prod_{i\in I} \mathrm{ev}_{P_i}, \mathrm{ev}_{P'}\right) \cong \left(\coprod_{i\in I} F(P_i)\right)(P),$$

we find that if F preserves coproducts then natural operations $\prod_{i\in I} \mathrm{ev}_{P_i} \to \mathrm{ev}_{P'}$ on F-models are generated by operations $\mathrm{ev}_{P_i} \to \mathrm{ev}_{P'}$ together with operations already defined for models of $\mathcal{P}$. In general, one may heuristically view $\mathcal{P}$-algebras as those theories obtained from $\mathcal{P}$ by adjoining only additional *unary* operations and relations.

All of this is useful for understanding the ordinary algebra of $\mathrm{Model}_F^{\heartsuit}$. However, our primary interest is in derived invariants; for instance, we would like to compute the Quillen cohomology of F-models. The first step in carrying out these computations is to understand abelianization for F-models, and here it is the right adjoint $F^{\vee}$ that makes algebras so useful. This right adjoint may be identified explicitly as $F^{\vee}(X)(P) = \mathrm{Hom}_{\mathcal{P}}(F(P), X)$; informally, $F^{\vee}$ is representable with representing object F. The key observation is that because $F^{\vee}$ preserves limits, it preserves all kinds of algebraic structure.

This plays out as follows. Write D for abelianization, so that $\mathcal{A}\mathrm{b}(\mathrm{Model}_{\mathcal{P}}^{\heartsuit}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{D\mathcal{P}}^{\heartsuit}$ and $\mathcal{A}\mathrm{b}(\mathrm{Model}_F^{\heartsuit}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{DF}^{\heartsuit}$. Then DF is an algebra over $D\mathcal{P}$, informally obtained by linearizing the unary operations used to form F. If B is an F-model, then we may consider the abelianizaton DB of its underlying $\mathcal{P}$-model; the action of F on B then linearizes to an action of DF on DB, and this provides a model for the abelianization of B as an F-model (Proposition 3.3). If F satisfies an additional smoothness condition, then all of this may be made to work for derived abelianization as well (Proposition 3.4). In this case, Quillen cohomology in $\mathcal{M}\mathrm{odel}_F$ is built in a simple way from Quillen homology in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ and the ordinary homological algebra of $\mathrm{L}\mathcal{M}\mathrm{od}_{DF}$. See Subsubsection 1.2.5.1 below for an example of this general recipe.

Section 3.1 covers some additional topics useful for working with theories of operations. Here we will just highlight one: in Subsection 3.1.5, we recall the concept of a *distributive law*, as discovered by Beck [**Bec69**]. These turn out to be the key to answering a number of basic questions that come up when working with theories. As one simple example, if $\mathcal{P}$ is a theory, F is a $\mathcal{P}$-algebra, and A is an F-model, then distributive laws allow one to understand the manner in which $\mathcal{A}\mathrm{b}(A/\mathcal{M}\mathrm{odel}_F^\heartsuit)$ is built from A and $\mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_F^\heartsuit)$.

1.2.4.2. *Plethories.* (Section 3.3).

Section 3.2 and Section 3.3 are essentially independent of each other, and we will begin by describing the latter, where we consider the special case of algebras over a theory of commutative rings. Let $\mathcal{P}$ be an additive symmetric monoidal theory (cf. Subsection 3.1.4), and let S be the free commutative monoid monad on $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^\heartsuit$, so that $\mathcal{M}\mathrm{odel}_{S\mathcal{P}}^\heartsuit \simeq \mathrm{C}\mathcal{M}\mathrm{on}(\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^\heartsuit)$. A $\mathcal{P}$-*plethory* is equivalent to the data of an $S\mathcal{P}$-algebra (Definition 3.9). These generalize the biring triples of Tall-Wraith [**TW70**] and plethories of Borger-Wieland [**BW05**]. Abelianization for rings over a plethory fits into the general story of abelianization for models over an algebra over a theory, and we unravel this more explicitly in Subsection 3.3.4.

One of the features that distinguishes rings over a plethory from more familiar algebraic structures is the presence of *nonlinear* structure. To successfully work with such plethories is to avoid dealing with this nonlinear structure by any means possible. It turns out that many plethories of interest are determined by their additive operations; here the classic example is the plethory of θ-rings, also known as δ-rings [**Joy85**]. In short, a θ-ring is an ordinary commutative ring R equipped with an operation $\theta\colon R \to R$ satisfying the identities that ensure $\psi(x) = x^p + p\theta(x)$ is generically a ring homomorphism. The operation θ is highly nonlinear, but it is determined by ψ if R is p-torsion free; as free θ-rings are p-torsion free, it follows that the entire concept of a θ-ring is encoded by the operation ψ together with knowledge that $\psi(x) \equiv x^p \pmod p$. This is a p-typical analogue of the Wilkerson criterion for lambda rings [**Wil82**, Proposition 1.2]. Work of Rezk [**Rez09**] shows that the theory of power operations for $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum of height h is similar in form to the theory of θ-rings, and this motivates studying the general situation.

Under a minor flatness assumption, it is possible to package together the additive operations of a plethory into a single algebraic object, using a generalization of the additive bialgebras considered in [**BW05**, Section 10], as follows. Let Λ be a $\mathcal{P}$-plethory, and write $\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit$ for the category of Λ-models, which we call Λ-rings. Given $P, P' \in \mathcal{P}$, write

$$\Gamma_{P,P'} = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit, \mathcal{A}\mathrm{b})}(\mathrm{ev}_P, \mathrm{ev}_{P'}).$$

This is the set of natural additive operations $\mathrm{ev}_P \to \mathrm{ev}_{P'}$ on Λ-rings, and defines the morphisms for a theory $\Gamma = \Gamma(\Lambda)$ with the same objects as $\mathcal{P}$. There is more structure present on Γ, encoding how these additive operations interact with the underlying $\mathcal{P}$-ring structure of Λ-ring. This extra structure may be summarized by saying that the forgetful functor $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit \to \mathrm{L}\mathcal{M}\mathrm{od}_\mathcal{P}^\heartsuit$ is equipped with the structure of a strong symmetric monoidal functor (Theorem 3.5), and we say that Γ is a $\mathcal{P}$-*cobialgebroid* (Definition 3.10). When $\mathcal{P}$ is the theory of ordinary commutative rings over a commutative ring R, then $\mathcal{P}$-cobialgebroids, i.e. strong symmetric monoidal functors to $\mathcal{M}\mathrm{od}_R^\heartsuit$ which are both monadic and comonadic, are equivalent to the twisted R-bialgebras used in [**BW05**] (Example 3.17). The abstract definition in terms of symmetric monoidal functors is more suitable for working over objects such as graded rings.

1.2.4.3. *Koszul algebras.* (Section 3.2).

The material described above serves to reduce various calculations to purely linear calculations. In Section 3.2, we turn to the topic of *additive theories*. The primary goal of this section is to develop a robust theory of *Koszul algebras*, in particular to gain access to their associated Koszul resolutions and Koszul complexes. Koszul algebras were first introduced by Priddy [**Pri70**], motivated by the example of the Steenrod algebra. A more general notion of Koszul algebra is necessary to incorporate examples of interest in homotopy theory. For example, many of the algebras derived from homotopy operations fail to contain their coefficient rings centrally; others fail to admit an augmentation; others may not even be algebras in the classical sense, perhaps due to the presence of instability conditions; and still others may be built out of richer algebraic objects, such as Mackey functors. In fact, all of these cases are encountered even when just considering analogues of the mod 2 Steenrod algebra.

These can all be encoded as algebras over an additive theory, and so we are left with developing a theory of Koszul algebras over additive theories. Fortunately, once the correct definition is known, the development is not significantly more difficult than the classical case. Let $\mathcal{P}$ be an additive theory, and let $F = \mathrm{colim}_{m\to\infty} F_{\leq m}$ be a filtered $\mathcal{P}$-algebra, in the sense that $F_{\leq 0}$ is the initial algebra and multiplication on on F restricts to $F_{\leq n} \circ F_{\leq m} \to F_{\leq n+m}$. To any filtered algebra F we may associate its associated graded algebra $\mathrm{gr}\, F$, and for any F-module M the monadic bar complex $C(F, F, M)$ (reviewed in Subsection 3.2.2) may be filtered in such a way that $\operatorname{gr} C(F,F,M) \cong C(\operatorname{gr} F, \operatorname{gr} F, \overline{M})$; here $\overline{M}$ is M considered as a $\operatorname{gr} F$-module via the augmentation on $\operatorname{gr} F$.

Definition (Definition 3.7). A filtered $\mathcal{P}$-algebra F is *Koszul* if:

(1) F and $\operatorname{gr} F$ are projective, in the sense that they restrict to functors $\mathcal{P} \to \mathcal{P}$;
(2) $H_n C(\operatorname{gr} F, \operatorname{gr} F, \overline{P})[m] = 0$ for $P \in \mathcal{P}$ and $n \neq m$. ◁

This definiton may be understood as follows. As $\operatorname{gr} F$ is augmented, one can define its homology $H_*(\operatorname{gr} F)$ and cohomology $H^*(\operatorname{gr} F)$ (Subsection 3.2.3). Any time that F is filtered and M is projective over $\mathcal{P}$, we may identify a subcomplex of $C(F,F,M)$ of the form $K(F,F,M) = FH_*(\operatorname{gr} F)(M) \subset C(F,F,M)$; the condition that F is Koszul implies that this inclusion is a quasiisomorphism (Theorem 3.1). Thus $K(F,F,M)$ is a small projective resolution of M. In particular, given any F-model N, one may form the Koszul complexes $K_F(M,N) = \mathrm{L}\mathcal{M}\mathrm{od}_F(K(F,F,M),N) \cong \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(M, H^*(\operatorname{gr} F)(N))$, and these provide small models for $\mathcal{E}\mathrm{xt}_F(M,N)$.

We describe these Koszul complexes explicitly in Subsection 3.2.6 and Subsection 3.2.7. As in the classic story, $\operatorname{gr} F$ is a quadratic algebra, and $H^*(\operatorname{gr} F)$ is its quadratic dual (Theorem 3.2). This describes the graded objects $K_F(M,N)$ and pairings between these, and the differential admits a simple description in terms of this (Theorem 3.3, Theorem 3.4).

1.2.5. Applications. (Chapter 4).

This chapter applies all the machinery of Chapter 2 and Chapter 3 to two examples.

1.2.5.1. $\mathbb{E}_\infty$ *algebras over* $\mathbb{F}_p$. (Section 4.1).

Let R be an $\mathbb{E}_\infty$ ring with $p = 0$ in $\pi_0 R$, and let $\mathbb{P}\mathcal{R} = \mathcal{C}\mathrm{Alg}_R^{\mathrm{free}}$ be the essential image of $\mathcal{M}\mathrm{od}_R^{\mathrm{free}}$ under the free functor $\mathbb{P}_R\colon \mathcal{M}\mathrm{od}_R \to \mathcal{C}\mathrm{Alg}_R$. Then $\mathcal{P}$ is a loop theory with $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \simeq \mathcal{C}\mathrm{Alg}_R$, and $\mathrm{h}\mathcal{P}$ is a theory of power operations for $\mathbb{E}_\infty$ algebras over R. There is a forgetful functor $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathbb{P}\mathcal{R}}^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$; this preserves colimits, and so realizes $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathbb{P}\mathcal{R}}^{\heartsuit} = \mathcal{R}\mathrm{ing}_{\mathrm{h}\mathbb{P}\mathcal{R}}^{\heartsuit}$ as the category of rings over an R_*-plethory. This is a consequence of the Künneth isomorphisms $\pi_*\mathbb{P}_R(F \oplus F') \cong \pi_*\mathbb{P}_R F \otimes_{R_*} \pi_*\mathbb{P}_R F'$, which hold for $F, F' \in \mathcal{M}\mathrm{od}_R^{\mathrm{free}}$ due to a stronger fact: each $\pi_*\mathbb{P}_R\Sigma^a R$ is free as an R_*-ring. This is well-known when $R = \mathbb{F}_p$, and in general the Hopkins-Mahowald Thom spectrum theorem may be used to produce an unstructured isomorphism $\pi_*\mathbb{P}_R\Sigma^a R \cong R_* \otimes \pi_*\mathbb{P}_{\mathbb{F}_p}\Sigma^a\mathbb{F}_p$ of rings.

Thus we are in a position to apply the machinery of Chapter 2 and Chapter 3. By way of example, the recipe discussed above in Subsubsection 1.2.4.1 amounts to the following (cf. Subsection 3.3.4). First, there are composition maps

$$\pi_c\mathbb{P}_R\Sigma^b R \times \pi_b\mathbb{P}_R\Sigma^a R \to \pi_c\mathbb{P}_R\Sigma^a R;$$

given $\alpha\colon \Sigma^c R \to \mathbb{P}_R\Sigma^b R$ and $\beta\colon \Sigma^b R \to \mathbb{P}_R\Sigma^a R$ in $\mathcal{M}\mathrm{od}_R$, their composition is the composite

$$m \circ \mathbb{P}_R\beta \circ \alpha\colon \Sigma^c R \to \mathbb{P}_R\Sigma^b R \to \mathbb{P}_R\mathbb{P}_R\Sigma^a R \to \mathbb{P}_R\Sigma^a R.$$

Each $\pi_*\mathbb{P}_R\Sigma^a R$ is augmented over R_*, and on indecomposables these compositions yield

$$Q(\pi_*\mathbb{P}_R\Sigma^b R)_c \otimes Q(\pi_*\mathbb{P}_R\Sigma^a R)_b \to Q(\pi_*\mathbb{P}_R\Sigma^a R)_c.$$

Let $\mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{\Delta(R)}$ be the category of R_*-modules M_* equipped with maps

$$Q(\pi_*\mathbb{P}_R\Sigma^a R)_b \otimes M_a \to M_b$$

satisfying the evident associativity and unitality conditions. Then there is an equivalence of categories $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}^{\heartsuit,\mathrm{aug}}_{\mathrm{h}\mathbb{P}\mathcal{R}}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{\Delta(R)}$. More generally, given $C_* \in \mathcal{R}\mathrm{ing}^{\heartsuit}_{\mathrm{h}\mathbb{P}\mathcal{R}}$, there is an equivalence $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}^{\heartsuit}_{\mathrm{h}\mathbb{P}\mathcal{R}}/C_*) \simeq \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{C_*\otimes_{R_*}\Delta(R)}$, where $C_* \otimes_{R_*} \Delta(R)$ is an object obtained from C_* and $\Delta(R)$ via a certain distributive law. This extends to an explicit description of the Quillen cohomology of $\mathrm{h}\mathbb{P}\mathcal{R}$-rings: given $A_* \to C_*$ in $\mathcal{R}\mathrm{ing}^{\heartsuit}_{\mathrm{h}\mathbb{P}\mathcal{R}}$ and $M_* \in \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{C_*\otimes_{R_*}\Delta(R)}$, there is an equivalence

$$\mathcal{H}_{\mathrm{h}\mathbb{P}\mathcal{R}/C_*}(A_*; M_*) \simeq \mathcal{E}\mathrm{xt}_{C_*\otimes_{R_*}\Delta(R)}(C_* \otimes^{\mathbb{L}}_{A_*} \mathbb{L}\Omega_{A_*|R_*}, M_*).$$

In the case where $A_* = \pi_*A$ and $C_* = \pi_*C$ for $A, C \in \mathcal{C}\mathrm{Alg}_R$, and $M_* = \pi_*\Omega^n C$ for $n \geq 1$, these are exactly the objects forming the layers of the filtration of $\mathcal{C}\mathrm{Alg}_R(A, C)$ given by Theorem 2.11.

Except for the homotopical input that $\pi_*\mathbb{P}_R\Sigma^a R$ is free as an R_*-ring, all of this is just a specialization of the general algebraic material that has already been discussed. In Section 4.1, we consider the particular case $R = \mathbb{F}_p$; here the structure of power operations is well-understood, and so there is more that can be said.

Write $\mathrm{DL} = \mathrm{h}\mathbb{P}\mathbb{F}_p$ for the plethory of power operations for $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$. We recall the structure of these power operations in Subsection 4.1.2; for instance, when $p = 2$, a DL-ring amounts to an $\mathbb{F}_{2*}$-ring A_* equipped with additive maps $Q^s\colon A_n \to A_{n+s}$ for $s \in \mathbb{Z}$, which are subject to certain explicitly describable Adem relations, Cartan formulas, and instability conditions. The category of DL-rings is closely related to the category of unstable rings over the Steenrod algebra, and we give the precise relation in Subsection 4.1.3.

As above, one may form an object Δ with $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}^{\heartsuit}_{\mathrm{DL}}/C_*) \simeq \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{C_*\otimes\Delta}$, and in this case Δ admits an explicit description in terms of generators and relations. The presence of instability conditions implies that Δ is not merely a $\mathbb{Z}$-graded $\mathbb{F}_p$-algebra, but it is an algebra for the theory of $\mathbb{F}_{p*}$-modules. The algebra Δ turns out to be Koszul, and we compute its cohomology in Subsection 4.1.5. This gives access to Koszul complexes for computing with Δ-modules which can be regarded as nonconnective analogues of the unstable Koszul complexes considered by Miller [Mil78].

In Subsection 4.1.6, we again elaborate on the mapping space obstruction theory of Theorem 2.11 in this context, and give some examples. In Subsection 4.1.7, we apply Theorem 2.7 to produce a generalization of the Basterra spectral sequence for computing the André-Quillen-Goodwillie towers of $\mathbb{E}_\infty$ rings in this context.

1.2.5.2. *Lubin-Tate spectra.* (Section 4.2).

We finally return to our original motivation in Section 4.2, where we give applications to $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum E of height h. In the end, this takes the same form as for $\mathbb{E}_\infty$ algebras in positive characteristic: there is an E_*-plethory $\mathbb{T}$ satisfying all the niceness properties one could hope for, and there are obstruction theories for working with $K(h)$-local $\mathbb{E}_\infty$ algebras over E with obstruction groups built from derived invariants of $\mathbb{T}$-rings. This turns out to not be obvious, due to some subtleties that arise in this context, but the preceding chapters have been written to handle this.

In the end, this works out as follows. Building on work of Strickland [**Str98**], which in turn builds on calculations of Kashiwabara [**Kas98**], Rezk [**Rez09**] has produced what is in our language an E_*-plethory $\mathbb{T}$ encoding the structure of E-power operations. In short, a $\mathbb{T}$-ring is very much like a θ-ring, only with more additive operations, which satisfy more complicated relations; moreover, $\mathbb{T}$-rings admit a conceptual interpretation in terms of the deformation theory of isogenies of formal groups. We recall all of this in Subsection 4.2.3. There are some additional properties special to $\mathbb{T}$; most notably, $\mathbb{T}$-rings are very close to being rings over an ungraded plethory. This is captured in [**Rez09**] using the notion of a twisted $\mathbb{Z}/(2)$-graded category, and in Subsection 4.2.1 we give a different packaging in terms of even-periodic plethories. All of these facts together reduce many computations with $\mathbb{T}$-rings to computations over an ungraded cobialgebroid Γ associated to $\mathbb{T}$; ungraded cobialgebroids are the coalgebraic analogue of formal category schemes, and we review this in Subsection 4.2.2.

Now let $\mathcal{P} = \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}$ be the category of $K(h)$-localizations of free $\mathbb{E}_\infty$ algebras over E. Then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \simeq \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$, but $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$ is not equivalent to the category of $\mathbb{T}$-rings: the $K(h)$-local condition enforces a completeness condition on homotopy groups, and this is not reflected in $\mathbb{T}$. We explain how to deal with this in Subsection 4.2.4; this goes as follows. Write $\mathfrak{m} \subset E_0$ for the maximal ideal of E_0. Then there is a category $\mathcal{M}\mathrm{od}_{E_*}^{\mathrm{Cpl}(\mathfrak{m})}$ of $\mathfrak{m}$-complete objects in the derived category of E_*-modules, and this is a localization of the dervied category of E_*-modules. Let $\mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m})}$ be the homotopy theory of simplicial $\mathbb{T}$-rings whose underlying object of $\mathcal{M}\mathrm{od}_{E_*}^{\mathrm{cn}}$ is $\mathfrak{m}$-complete; this is a localization of $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$. Then there is an equivalence of homotopy theories $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} \simeq \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m})}$ (Theorem 4.7). In particular, the Quillen cohomology of a model of $\mathrm{h}\mathcal{P}$ agrees with the Quillen cohomology of its underlying $\mathbb{T}$-ring.

Thus the material of Chapter 2 indeed provides obstruction theories for working with $\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ with obstruction groups built from derived invariants of $\mathbb{T}$-rings. These turn out to be very pleasant at heights $h \leq 2$, as a consequence of the following. Let Δ be the E_*-algebra such that $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{aug},\heartsuit}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_\Delta$. A theorem of Rezk [**Rez12**] shows that Δ is a Koszul algebra, and moreover $H^n(\Delta) = 0$ for $n > h$. The theory of Koszul resolutions then implies that $\mathrm{Ext}^n_\Delta(M,N) = 0$ for $n > 0$ whenever M is a Δ-module which is projective as an E_*-module. With a bit more work, all of this may be made to work for other slices of $\mathcal{R}\mathrm{ing}_{\mathbb{T}}$, and to play well with completions; as a sample of the sort of subtlety that must be dealt with, observe that a Δ-module whose underlying E_*-module is the completion of a projective E_*-module need not be the completion of a Δ-module whose underlying E_*-module is projective.

Once all the details are dealt with, we are placed in a good position to give applications. In Subsection 4.2.5, we unravel what the mapping space obstruction theory of Theorem 2.11 says in this context, and in Subsection 4.2.6, we discuss the topological André-Quillen homology and cohomology of $K(h)$-local $\mathbb{E}_\infty$ algebras over E. One easily stated special case of the mapping space obstruction theory is the following.

Theorem (Theorem 4.8). Let E be a Lubin-Tate spectrum of height $h \leq 2$, and fix $A, B \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$. Suppose:

(1) A_* and B_* are concentrated in even degrees;
(2) A_0 is the completion of a localization of a polynomial ring over E_0.

Then every $\mathbb{T}$-ring map $A_* \to B_*$ lifts to a map $A \to B$ in $\mathcal{C}\mathrm{Alg}_E$, uniquely if $h = 1$. ◁

This follows quickly from our general machinery combined with bounds on certain Ext groups implied by Rezk's theorem on Δ, and the existence of $\mathbb{E}_\infty$ orientations at heights $h \leq 2$ is an immediate corollary (Theorem 4.9).

CHAPTER 2

Homotopy

2.1. Mal'cev theories

This section covers the general properties of Mal'cev theories. In particular, in Subsection 2.1.1, we show that if $\mathcal{P}$ is a Mal'cev theory, then the category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ of models of $\mathcal{P}$ freely adjoins geometric realizations to $\mathcal{P}$. Moreover, we verify that $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is presentable under some minor smallness conditions on $\mathcal{P}$. In Subsection 2.1.2, we restrict to the case where $\mathcal{P}$ is a discrete Mal'cev theory, verify that $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is the underlying ∞-category of Quillen's model structure on simplicial objects in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, and review the resulting notion of left-derived functor.

2.1.1. Definitions and universal properties. The structure of a *herd* on a set X is a ternary operation t satisfying $t(x,x,y)=y$ and $t(x,y,y)=x$; write $\mathcal{H}\mathrm{rd}$ for the category of herds. Herds are the models of a finitary and discrete algebraic theory, so herd objects can be defined in an arbitrary category with finite products, allowing for the following definition.

Definition 2.1. A *Mal'cev theory* is a category $\mathcal{P}$ such that

(1) $\mathcal{P}$ admits all small coproducts;
(2) All objects of $\mathcal{P}$ admit the structure of a coherd.

For a regular cardinal κ, a Mal'cev theory $\mathcal{P}$ is said to be *κ-bounded* if there exists a small full subcategory $\mathcal{P}_0 \subset \mathcal{P}$ such that

(3) $\mathcal{P}_0$ is closed under κ-small coproducts;
(4) Every object of $\mathcal{P}$ is a retract of a small coproduct of objects of $\mathcal{P}_0$;
(5) For every $P \in \mathcal{P}_0$ and every set of objects $\{P_i' : i \in I\}$ in $\mathcal{P}$, the canonical map $\operatorname{colim}_{F\subset I,\,|F|<\kappa} \mathrm{Map}_{\mathcal{P}}(P, \coprod_{i\in F} P_i') \to \mathrm{Map}_{\mathcal{P}}(P, \coprod_{i\in I} P_i')$ is an equivalence.

A Mal'cev theory $\mathcal{P}$ is *bounded* if it is κ-bounded for some κ, and is *discrete* if it is a 1-category. $\triangleleft$

We will refer to Mal'cev theories as just *theories*. After this subsection, we will moreover assume that all of our theories are bounded; see Remark 2.3. Throughout this subsection, and everywhere else in the thesis, $\mathcal{P}$ will always refer to a theory, at times satisfying additional assumptions. In general we will write, for instance, $\mathrm{Map}_{\mathcal{P}}$ rather than $\mathrm{Map}_{\mathcal{M}\mathrm{odel}_{\mathcal{P}}}$.

Lemma 2.1.

(1) Every map of simplicial herds which is a degreewise surjection onto path components in its image is a Kan fibration;
(2) Every group object in $\mathcal{H}\mathrm{rd}$ is abelian;
(3) Every covering map in the category of herd objects in $\mathcal{G}\mathrm{pd}_\infty$ which admits a section has trivial monodromy.

Proof. (1) This is well known should we replace herds with groups, and the same proof applies. In brief, suppose given a surjection $p\colon E \to B$ of simplicial herds, elements $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in E_n$ such that $d_i(x_j) = d_{j-1}(x_i)$ for $i < j$ and $i \neq k$, and $y \in B_{n+1}$ such that $d_i(y) = p(x_i)$ for $i \neq k$. Inductively define $w_r \in E_{n+1}$ such that $p(w_r) = y$ and $d_i w_r = x_i$ for $i \leq r$ and $i \neq k$ by choosing w_{-1} to be any element in the preimage of y, and setting $w_r = t(w_{r-1}, s_r d_r w_{r-1}, s_r x_r)$, except when $r = k$, in which case $w_r = w_{r-1}$. Then $w_{n+1} \in E_{n+1}$ witnesses the Kan condition.

(2) Observe that if G is a group object in $\mathcal{H}\mathrm{rd}$ with unit e, then for any $g, h \in G$ we have $gh = t(g,e,e)t(e,e,h) = t(g,e,h) = t(e,e,h)t(g,e,e) = hg$.

(3) Let $\pi\colon E \to X$ be a covering map of herd objects in $\mathcal{G}\mathrm{pd}_\infty$ with section $s\colon X \to E$. To show that this cover has trivial monodromy, it is sufficient to verify that for all $x \in X$ the inclusion $\pi^{-1}(x) \to E$ admits a retraction. Such a retraction is given by $e \mapsto t(e, s\pi(e), s(x))$. □

Definition 2.2. The category of *models* of a theory $\mathcal{P}$ is the full subcategory $\mathcal{M}\mathrm{odel}_\mathcal{P} \subset \mathrm{Psh}(\mathcal{P})$ of small presheaves X on $\mathcal{P}$ such that for any set $\{P_i : i \in I\}$ of objects in $\mathcal{P}$, the canonical map

$$X\left(\coprod_{i\in I} P_i\right) \to \prod_{i\in I} X(P_i)$$

is an equivalence. The category of *discrete models* of $\mathcal{P}$ is the full subcategory $\mathcal{M}\mathrm{odel}_\mathcal{P}^\heartsuit \subset \mathcal{M}\mathrm{odel}_\mathcal{P}$ of $\mathcal{S}\mathrm{et}$-valued models of $\mathcal{P}$. ◁

As $\mathcal{P}$ consists of coherds, if $X \in \mathcal{M}\mathrm{odel}_\mathcal{P}$ and $P \in \mathcal{P}$, then $X(P)$ is itself a herd. This need not be natural in P, but it is natural in X, i.e. maps $X \to Y$ of models induce maps $X(P) \to Y(P)$ of herds. This is sufficient for the following.

Proposition 2.1. The subcategory $\mathcal{M}\mathrm{odel}_\mathcal{P} \subset \mathrm{Psh}(\mathcal{P})$ is closed under small limits and geometric realizations.

Proof. The assertion regarding small limits is clear, so we must verify that the pointwise geometric realization of a simplicial object in $\mathcal{M}\mathrm{odel}_\mathcal{P}$ again lives in $\mathcal{M}\mathrm{odel}_\mathcal{P}$. As $\mathcal{P}$ consists of coherds and the forgetful functor $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty) \to \mathcal{G}\mathrm{pd}_\infty$ preserves geometric realizations, it is sufficient to verify that geometric realizations preserve small products in the category $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty)$ of herd objects in $\mathcal{G}\mathrm{pd}_\infty$. As $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty)$ is modeled by the model category of simplicial herds, we can model a simplicial object in $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty)$ by a bisimplicial herd and its geometric realization by the diagonal of this bisimplicial herd. As all simplicial herds are fibrant by Lemma 2.1, small products of simplicial herds are homotopy products, so the result follows as products and diagonals of bisimplicial sets commute. $\square$

We also record the following here.

Lemma 2.2. Fix a levelwise Cartesian square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ Y \arrow[r] & Z \end{tikzcd}$$

of simplicial objects in $\mathcal{M}\mathrm{odel}_\mathcal{P}$, and suppose that π is levelwise a π_0-surjection. Then the square remains Cartesian after geometric realization.

Proof. By Proposition 2.1, we may by evaluating on $P \in \mathcal{P}$ reduce to proving the corresponding statement with $\mathcal{M}\mathrm{odel}_\mathcal{P}$ replaced by $\mathcal{H}\mathrm{rd}(\mathcal{G}\mathrm{pd}_\infty)$. The square can now be modeled as a Cartesian square of bisimplicial herds in which the map π is levelwise a surjection. This square remains Cartesian upon taking diagonals, and remains homotopy Cartesian as π remains a Kan fibration by Lemma 2.1. This proves the claim. $\square$

Remark 2.1. After this point, herds will no longer appear explicitly. $\triangleleft$

Observe that $\mathcal{M}\mathrm{odel}_\mathcal{P}$ consists of those small presheaves on $\mathcal{P}$ which are local with respect to the class of maps of the form $\coprod_{i\in I} h(P_i) \to h(\coprod_{i\in I} P_i)$ for $\{P_i : i \in I\}$ a set of objects in $\mathcal{P}$.

Lemma 2.3. The inclusion $R\colon \mathcal{M}\mathrm{odel}_\mathcal{P} \to \mathrm{Psh}(\mathcal{P})$ admits a left adjoint.

Proof. By the Yoneda lemma, it is sufficient to verify the pointwise assertion that for all $X \in \mathrm{Psh}(\mathcal{P})$, the functor $\mathrm{Map}_{\mathrm{Psh}(\mathcal{P})}(X, R(-))\colon \mathcal{M}\mathrm{odel}_\mathcal{P} \to \mathcal{G}\mathrm{pd}_\infty$ is representable; see for instance [**Cis19**, Proposition 6.1.11]. By definition of $\mathrm{Psh}(\mathcal{P})$, the presheaf X is small, and thus admits a presentation of the form $X \simeq \mathrm{colim}_{n\in\Delta^{\mathrm{op}}} \coprod_{i\in I_n} h(P_{n,i})$ for some sets I_n and $P_{n,i} \in \mathcal{P}$. As a consequence, we have

$$\mathrm{Map}_{\mathrm{Psh}(\mathcal{P})}(X, R(-)) \simeq \mathrm{Map}_{\mathrm{Psh}(\mathcal{P})}(\operatorname*{colim}_{n\in\Delta^{\mathrm{op}}} h(\coprod_{i\in I_n} P_{n,i}), R(-)).$$

We conclude by Proposition 2.1, which shows that $\mathrm{colim}_{n\in\Delta^{\mathrm{op}}} h(\coprod_{i\in I_n} P_{n,i})$ lives in $\mathcal{M}\mathrm{odel}_\mathcal{P}$. $\square$

Proposition 2.2. For a theory $\mathcal{P}$,

(1) The category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ admits all small limits and colimits;
(2) The subcategory $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \subset \mathrm{Psh}(\mathcal{P})$ is the smallest full subcategory containing all representables and closed under geometric realizations;
(3) For $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, the functor $\mathrm{Map}_{\mathcal{P}}(X, -)$ preserves geometric realizations if and only if X is a retract of a representable.

Proof. (1) This follows immediately from Lemma 2.3, as $\mathrm{Psh}(\mathcal{P})$ admits all small colimits and limits and a reflective subcategory of a category admitting all small limits and colimits admits the same.

(2) This follows from the proof of Proposition 2.3, which gives a way of writing any $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ as a geometric realization of representables.

(3) Note that if X is a retract of a representable, then $\mathrm{Map}_{\mathrm{Psh}(\mathcal{P})}(X, -)$ preserves all small colimits, so $\mathrm{Map}_{\mathcal{P}}(X, -)$ preserves all geometric realizations, as these are computed in $\mathrm{Psh}(\mathcal{P})$. Conversely, if $\mathrm{Map}_{\mathcal{P}}(X, -)$ preserves geometric realizations, then upon using (2) to write $X \simeq \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} h(P_n)$, we find $\mathrm{Map}_{\mathcal{P}}(X, X) \simeq \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{Map}_{\mathcal{P}}(X, h(P_n))$, so that the identity of X factors through some representable. $\square$

Theorem 2.1. Let $\mathcal{D}$ be a category admitting geometric realizations, and let $F\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{D}$ be a functor. Write $f = F \circ h\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{D}$. Then

(1) F preserves geometric realizations if and only if it arises as the left Kan extension of f along $h\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$;
(2) F preserves colimits if and only if F preserves geometric realizations and f preserves coproducts;
(3) If the following hold, then F is fully faithful:
    (a) F preserves geometric realizations,
    (b) f is fully faithful,
    (c) For all $P \in \mathcal{P}$, the functor $\mathcal{D}(f(P), -)$ preserves geometric realizations;
(4) If the following hold, then F is an equivalence:
    (d) F preserves colimits,
    (e) F is fully faithful,
    (f) The right adjoint to F, given by $G(D) = \mathcal{D}(f(-), D)$, is conservative.

Proof. (1–2) These follow quickly from Proposition 2.2 and the general theory of cocompletions of categories, as from [**Lur17b**, Section 5.3.6].

(3) Suppose given $F\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{D}$ satisfying conditions (a)-(c). We must show that

$$\mathrm{Map}_{\mathcal{P}}(X, Y) \simeq \mathrm{Map}_{\mathcal{D}}(F(X), F(Y)).$$

As X may be written as a geometric realization of representable functors, by (a) we reduce to showing that

$$\mathrm{Map}_{\mathcal{P}}(h(P), Y) \simeq \mathrm{Map}_{\mathcal{D}}(f(P), F(Y))$$

for $P \in \mathcal{P}$. As Y may also be written as a geometric realization of representable functors, by (a) and (c) we reduce to showing that

$$\mathrm{Map}_{\mathcal{P}}(h(P), h(P')) \simeq \mathrm{Map}_{\mathcal{D}}(f(P), f(P'))$$

for $P, P' \in \mathcal{P}$, which is a consequence of (b).

(4) Condition (d) ensures that the functor G described in (f) is right adjoint to F, and the assertion then follows from the general fact that an adjunction $F \dashv G$ with F fully faithful and G conservative is an equivalence. □

Suppose now that $\mathcal{P}$ is κ-bounded, choose a subcategory $\mathcal{P}_0 \subset \mathcal{P}$ realizing this, and let $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0) \subset \mathrm{Psh}(\mathcal{P}_0)$ be the full subcategory consisting of presheaves which preserve κ-small products.

Lemma 2.4.

(1) The subcategory $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0) \subset \mathrm{Psh}(\mathcal{P}_0)$ is closed under geometric realizations and κ-filtered colimits;

(2) The category $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$ consists of those objects of $\mathrm{Psh}(\mathcal{P}_0)$ local with respect to the set of maps of the form $\coprod_{i\in F} h(P_i) \to h(\coprod_{i\in I} P_i)$ where $\{P_i : i \in F\}$ is a set of objects of $\mathcal{P}_0$ with $|F| < \kappa$.

In particular, $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$ is a κ-compactly generated presentable category. □

Proposition 2.3. Restriction $R\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$ is an equivalence.

Proof. We verify the conditions of Theorem 2.1. As geometric realizations are computed pointwise in either category, they are preserved by R. Next, by our smallness assumption on the objects of $\mathcal{P}_0$, we find that for any collection of objects $\{P_i : i \in I\}$ in $\mathcal{P}_0$ there is an equivalence

$$R(h(\coprod_{i\in I} P_i)) \simeq \operatorname*{colim}_{\substack{F\subset I\\ |F|<\kappa}} h(\coprod_{i\in F} P_i) \simeq \operatorname*{colim}_{\substack{F\subset I\\ |F|<\kappa}} \coprod_{i\in F} h(P_i) \simeq \coprod_{i\in F} h(P_i),$$

and hence $\mathcal{P} \to \mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$ preserves coproducts. It follows that for any $P \in \mathcal{P}$, the functor $\mathrm{Map}_{\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)}(h(P), -)$ preserves geometric realizations. The right adjoint to R is conservative, so the conditions of Theorem 2.1 are satisfied. □

Remark 2.2. Everything in this subsection has a fully 1-categorical analogue, where all categories are taken to be 1-categories, $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is replaced by $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, and geometric realizations reduce to reflexive coequalizers. ◁

Remark 2.3. We assume for the rest of this thesis that all theories are bounded. In order to avoid cumbersome notation, we will adopt the following convention: if $\mathcal{P}$ is a κ-bounded theory, choose $\mathcal{P}_0 \subset \mathcal{P}$ realizing this; now, $\mathrm{Psh}(\mathcal{P})$ refers to $\mathrm{Psh}(\mathcal{P}_0)$, $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ refers to $\mathrm{Psh}^{\Pi_\kappa}(\mathcal{P}_0)$, and so forth. In case one should meet a theory which is not bounded, we point out that an arbitrary theory $\mathcal{P}$ is of the form $\mathcal{P} = \mathcal{P}'_0$ where $\mathcal{P}'$ is a bounded theory with respect to a larger universe. ◁

2.1.2. Rigidification and left-derived functors. Throughout this subsection, all of our theories are assumed to be discrete theories. In this subsection, we briefly review the notion of left-derived functors available for categories of the form $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. This story is classical, and goes back to [**Qui67**] and [**DP61**]; see also [**TV69**]. To facilitate comparisons with the classical theory, we begin with an identification of a model for $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

For a category $\mathcal{C}$, write $\mathrm{s}\mathcal{C} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ for the category of simplicial objects in $\mathcal{C}$.

Lemma 2.5 ([**Qui67**, Section II.4]). There is a simplicial model structure on $\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ in which a map $f\colon X \to Y$ is a weak equivalence, resp., fibration, if and only if for all $P \in \mathcal{P}$ the map $f(P)\colon X(P) \to Y(P)$ is a weak equivalence, resp., fibration. □

Theorem 2.2. The (∞-categorical) colimit functor

$$C\colon \mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{M}\mathrm{odel}_{\mathcal{P}}) \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$$

realizes $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ as the underlying ∞-category of $\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$.

Proof. Let W denote the class of weak equivalences in $\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. Then we need to show the following:

(1) C inverts W;
(2) C is essentially surjective;
(3) For $X, Y \in \mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ with X cofibrant, C induces an equivalence $\underline{\mathrm{Map}}_{\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}}(X, Y) \simeq \mathrm{Map}_{\mathcal{P}}(CX, CY)$, where $\underline{\mathrm{Map}}_{\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}}$ denotes the simplicial enrichment of $\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$.

These are themselves consequences of the following observations:

(a) For $X \in \mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ and $P \in \mathcal{P}$, there is an equivalence

$$X(P) \simeq \underline{\mathrm{Map}}_{\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}}(h(P), X) \simeq \mathrm{Map}_{\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}[W^{-1}]}(h(P), X);$$

(b) As homotopy geometric realizations in $\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ are modeled by diagonals, for $X \in \mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ there is an equivalence

$$X \simeq \operatorname*{hocolim}_{n \in \Delta^{\mathrm{op}}} X_n;$$

(c) For $X \in \mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ and $P \in \mathcal{P}$, as $\underline{\mathrm{Map}}_{\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}}(h(P), X)_n \cong X_n(P)$, there is an equivalence

$$\underline{\mathrm{Map}}_{\mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}}(h(P), X) \simeq \operatorname*{hocolim}_{n \in \Delta^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{P}}(h(P), X_n).$$

$\square$

We now review the relevant notion of left-derived functor. Let $\mathcal{P}$ and $\mathcal{P}'$ be discrete theories, and fix an arbitrary functor $\overline{f} \colon \mathcal{P}' \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. By left Kan extension, we obtain a functor $\overline{F} \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ preserving reflexive coequalizers. By left Kan extension of the composite $f \colon \mathcal{P}' \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, we obtain a functor $f_! \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ preserving geometric realizations such that $\pi_0 f_! X = \overline{F} X$ for any $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit}$.

Proposition 2.4. Fix notation as above. Fix $X' \in \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$, and choose some $X'_\bullet \in \mathrm{s}\mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit}$ modeling X. Choose a simplicial object $P'_\bullet$ of $\mathcal{P}'$ together with a weak equivalence $h(P'_\bullet) \to X'_\bullet$. Then $f_! X'$ is modeled by $f P'_\bullet$.

Proof. By Theorem 2.2, to say that $X'_\bullet$ models X is to say we have chosen an identification $\mathrm{colim}_{n \in \Delta^{\mathrm{op}}} X'_n = X'$ in $\mathcal{M}\mathrm{odel}_{\mathcal{P}'}$, and to say $h(P'_\bullet) \to X'_\bullet$ is a weak equivalence is to say it induces $\mathrm{colim}_{n \in \Delta^{\mathrm{op}}} h(P'_n) \simeq \mathrm{colim}_{n \in \Delta^{\mathrm{op}}} X'_n \simeq X'$ in $\mathcal{M}\mathrm{odel}_{\mathcal{P}'}$. By definition of $f_!$, we learn

$$f_! X' \simeq f_! \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} h(P'_n) \simeq \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} f P'_n,$$

and the result follows as $\mathrm{colim}_{n \in \Delta^{\mathrm{op}}} f P'_n$ is modeled by $f P'_\bullet$. $\square$

This justifies writing $\mathbb{L}\overline{F} = f_! \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ and calling it the total left-derived functor of $\overline{F}$, for by Proposition 2.4 this is equivalent to any other correct definition of $\mathbb{L}\overline{F}$.

2.2. Loop theories

This section covers some generalities of loop theories. The basic example is the category $\mathcal{P} = \mathcal{M}\mathrm{od}_R^{\mathrm{free}}$ of free R-modules for some $\mathbb{A}_\infty$-ring R; here we allow "free R-module" to include suspensions and desuspensions of R. If M is an R-module, then $h(M) \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ lives in the full subcategory $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ consisting of those X with the additional property that $X(\Sigma F) \simeq \Omega X(F)$, and this turns out be a full characterization, i.e. there is an equivalence $\mathcal{M}\mathrm{od}_R \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$; a particular case of this appears in [**HL17**, Proposition 4.2.5]. Loop theories axiomatize the general situation.

After giving some definitions and fixing some notation in Subsection 2.2.1, in Subsection 2.2.2 we verify that the spiral sequence, as interpreted in [**Pst17**], holds equally well in our setting; this is the main tool for relating $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ to $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$. In Subsection 2.2.3, we record some tools for writing categories $\mathcal{M}$ as $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ for some $\mathcal{P} \subset \mathcal{M}$, and for describing the categories $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$.

2.2.1. Definitions and notation. Fix a theory $\mathcal{P}$.

Definition 2.3. The theory $\mathcal{P}$ is:

(1) A *loop theory* if for all finite wedges of spheres F and all $P \in \mathcal{P}$, the tensor $F \otimes P = \operatorname{colim}_{x \in F} P$ exists in $\mathcal{P}$;
(2) A *pointed loop theory* if moreover $\mathcal{P}$ is pointed and admits suspensions;
(3) An *additive loop theory* if it is pointed and additive;
(4) A *stable loop theory* if it is pointed and $\Sigma\colon \mathcal{P} \to \mathcal{P}$ is an equivalence. ◁

Suppose now that $\mathcal{P}$ is a loop theory, and define $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ to consist of those X satisfying the additional condition that $X(F \otimes P) \simeq X(P)^F$ for all $P \in \mathcal{P}$ and finite wedge of spheres F. In other words, $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is the full subcategory of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ local with respect to $F \otimes h(P) \to h(F \otimes P)$ for all $P \in \mathcal{P}$ and finite wedge of spheres F. Because we are assuming that $\mathcal{P}$ is bounded, as laid out in Remark 2.3, we obtain the following.

Lemma 2.6. The category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is an accessible localization of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$. In particular, it is presentable. □

Remark 2.4. We do not know a good description of the localization $L\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ in general, but we will be able to give more information in the stable case (Theorem 2.6). ◁

For $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ and F a finite wedge of spheres, write X_F for the model of $\mathcal{P}$ defined by $X_F(P) = X(F \otimes P)$. Then there are canonical maps $X_F \to X^F$, and the condition that $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is equivalent to the condition that these maps be equivalences for all F. It turns out to be sufficient to verify this when $F = S^1$.

Lemma 2.7. Fix a coCartesian diagram

$$\begin{tikzcd} F_1 & F_2 \arrow[l] \\ F_3 \arrow[u] & F_4 \arrow[l] \arrow[u, "i"'] \end{tikzcd}$$

of wedges of spheres, and fix $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$. Then the resulting square

$$\begin{tikzcd} X_{F_1} \arrow[r] \arrow[d] & X_{F_2} \arrow[d] \\ X_{F_3} \arrow[r] & X_{F_4} \end{tikzcd}$$

is Cartesian. In particular, the cogroup structure on S^n gives maps

$$X_{S^n} \to X_{S^n \vee S^n} \simeq X_{S^n} \times_X X_{S^n}$$

making X_{S^n} into a group object in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}/X$ for $n \geq 1$.

Proof. If X is representable, or more generally if $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, then this is clear. In general, by splitting off the path components not in the image of i, we may reduce to the case where i is an injection on path components. Here the claim follows by writing X as a geometric realization of representables and appealing to Lemma 2.2. $\square$

Proposition 2.5. Fix $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$. Then the following are equivalent:

(1) $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$;
(2) The map $X_{S^1} \to X^{S^1}$ is an equivalence.

Proof. The implication (1) $\Rightarrow$ (2) is clear, so suppose conversely that $X_{S^1} \to X^{S^1}$ is an equivalence. We must show that $X_F \to X^F$ is an equivalence when F is any finite wedge of spheres. By Lemma 2.7, there are natural equivalences $X_{F' \vee F''} \simeq X_{F'} \times_X X_{F''}$, so we can reduce to $F = S^n$. For $n = 0$, this is a consequence of the fact that $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$. For $n \geq 2$, that $X_{S^n} \to X^{S^n}$ is an equivalence follows from an inductive argument using the Cartesian squares

$$\begin{tikzcd} X_{S^{n+1}} \arrow[r] \arrow[d] & (X_{S^n})_{S^1} \arrow[d] \\ X \arrow[r] & X_{S^n} \times_X X_{S^1} \end{tikzcd}$$

for $n \geq 1$, obtained by applying Lemma 2.7 to the cofibering $S^n \vee S^1 \to S^n \times S^1 \to S^{n+1}$ and identifying $X_{S^n \times S^1} \simeq (X_{S^n})_{S^1}$. $\square$

We end this subsection by introducing some additional notation. When $\mathcal{P}$ is pointed, write X_{Σ^n} for the presheaf $X_{\Sigma^n}(P) = X(\Sigma^n P)$. If, for instance, $\mathcal{P}$ is additive, then one may split $X_{S^n} \simeq X \times X_{\Sigma^n}$, so in particular $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ consists of those $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ such that $X_\Sigma \simeq \Omega X$.

The functor $P \mapsto S^n \otimes P$ descends to a functor on $\mathrm{h}\mathcal{P}$; for $X \in \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$, write $X\langle n \rangle$ for the restriction of X along this functor. Thus $\pi_0(X_{S^n}) = (\pi_0 X)\langle n \rangle$ for $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$. Similarly, when $\mathcal{P}$ is pointed, write $X[n]$ for the restriction of X along the functor on $\mathrm{h}\mathcal{P}$ obtained from $\Sigma^n \colon \mathcal{P} \to \mathcal{P}$. We point out that these constructions are not intrinsic to the theory $\mathrm{h}\mathcal{P}$, but rely on extra structure coming from $\mathcal{P}$.

2.2.2. The spiral. Let $\mathcal{P}$ be a loop theory, and write $\tau \colon \mathcal{P} \to \mathrm{h}\mathcal{P}$ for the canonical map to its homotopy category. To connect $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ with $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$, we will need to understand $\tau_! \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$. This understanding is achieved via the following.

Theorem 2.3 ([**Pst17**, Theorem 2.86]). For $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$,

(1) The map $X \to \tau^* \tau_! X$ is a π_0-equivalence;
(2) There is a natural Cartesian square

$$\begin{tikzcd} B_X X_{S^1} \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & \tau^* \tau_! X \end{tikzcd},$$

where $B_X X_{S^1}$ is the delooping of X_{S^1} in the slice category $\mathcal{M}\mathrm{odel}_{\mathcal{P}}/X$.

Proof. When $X = h(P)$ with $P \in \mathcal{P}$, as $\tau_! h(P) = h(\tau P)$ it follows that $\tau^* \tau_! X = \pi_0 X$. In this case, $X \to \tau^* \tau_! X$ is certainly a π_0-equivalence, and the above square becomes

$$\begin{tikzcd} B_X X^{S^1} \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & \pi_0 X \end{tikzcd},$$

which is Cartesian. Next, observe that the terms in the original square, as well as the property of $X \to \tau^* \tau_! X$ being a π_0-equivalence, are compatible with the formation of geometric realizations. By writing X as a geometric realization of representables, we may conclude with an application of Lemma 2.2. $\square$

The following is sufficient for many applications.

Corollary 2.1. For $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, the map $\tau_! X \to \pi_0 X$ is an equivalence if and only if $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$.

Proof. Fix $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, and consider the cube

$$\begin{tikzcd} B_X X_{S^1} \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dr] \arrow[dd] & \\ & B_X X^{S^1} \arrow[rr] \arrow[dd] & & X \arrow[dd] \\ X \arrow[rr] \arrow[dr] & & \tau^* \tau_! X \arrow[dr] & \\ & X \arrow[rr] & & \pi_0 X \end{tikzcd}$$

in which the front and back faces are Cartesian. If $\tau^* \tau_! X \to \pi_0 X$ is an equivalence, then $B_X X_{S^1} \to B_X X^{S^1}$ must be an equivalence. Conversely, if $B_X X_{S^1} \to B_X X^{S^1}$ is an equivalence, then as $\tau^* \tau_! X \to \pi_0 X$ is a π_0-equivalence, the right square is Cartesian, and this implies that $\tau^* \tau_! X \simeq \pi_0 X$. $\square$

2.2.3. Producing examples. This subsection is concerned with producing and identifying categories of the form $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, as well as their associated algebraic categories $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \simeq \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$. See Subsection 1.2.5 for some explicit examples.

A simple class of examples is given by the following observation: if $\mathcal{P}$ is a discrete theory, then $\mathcal{P}$ is a loop theory with $F \otimes P \simeq (\pi_0 F) \otimes P$ for F a finite wedge of spheres and $P \in \mathcal{P}$. In this case, $\pi_0 \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is an equivalence.

More interesting examples come from loop theories with more homotopical structure. In [**Pst17**, Proposition 3.1], the following example is given: if $\mathcal{P} \subset \mathcal{G}\mathrm{pd}_{\infty}^{*}$ is the full subcategory of wedges of positive-dimensional spheres, then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is the category of pointed connected ∞-groupoids, and $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is the category of Π-algebras. We are particularly interested in examples arising from spectral algebra, so we instead begin with the stable case.

Lemma 2.8. If $\mathcal{P}$ is a stable loop theory, then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is a stable category.

Proof. This is a consequence of [**Lur17a**, Corollary 1.4.2.27], as precomposition with the equivalence $\Sigma \colon \mathcal{P} \to \mathcal{P}$ agrees with Ω on $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$. □

Theorem 2.4. Let $\mathcal{M}$ be a stable category admitting small colimits, and let $\mathcal{P} \subset \mathcal{M}$ be a full subcategory which is a stable loop theory closed under coproducts and suspensions in $\mathcal{M}$. Then

(1) The restricted Yoneda embedding $h \colon \mathcal{M} \to \mathrm{Psh}(\mathcal{P})$ is fully faithful upon restriction to the thick subcategory generated by $\mathcal{P}$;
(2) The restricted Yoneda embedding yields an equivalence $\mathcal{M} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ provided either of the following is satisfied:
   (a) The restricted Yoneda embedding is conservative and $\mathcal{P}$ is generated under coproducts by objects which are compact in $\mathcal{M}$;
   (b) There is a fixed finite diagram $\mathcal{J}$ such that every object of $\mathcal{M}$ may be written as a $\mathcal{J}$-shaped colimit of objects of $\mathcal{P}$.

Proof. (1) Write $k \colon \mathcal{M} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$. As k is a limit-preserving functor between stable categories, k preserves finite colimits. Fix $Y \in \mathcal{M}$. Then the collection of $X \in \mathcal{M}$ such that

$$\mathrm{Map}_{\mathcal{M}}(X, Y) \to \mathrm{Map}_{\mathcal{P}}(k(X), k(Y))$$

is an equivalence is a thick subcategory of $\mathcal{M}$ containing $\mathcal{P}$, proving (1).

(2) First we claim that in either case k preserves all colimits. In case (a), k preserves filtered colimits, so this follows from preservation of finite colimits. In case (b), it is sufficient to verify that k preserves coproducts. Given a collection $\{M_i : i \in I\}$ of objects of $\mathcal{M}$, we may write $M_i \simeq \operatorname{colim}_{j \in \mathcal{J}} M_{i,j}$, and so compute

$$k\left(\bigoplus_{i \in I} M_i\right) \simeq k\left(\operatorname*{colim}_{j \in \mathcal{J}} \bigoplus_{i \in I} P_{i,j}\right) \simeq \operatorname*{colim}_{j \in \mathcal{J}} \bigoplus_{i \in I} k\left(P_{i,j}\right) \simeq \bigoplus_{i \in I} k\left(M_i\right),$$

as k preserves all finite colimits and all small coproducts of objects of $\mathcal{P}$.

As $\mathcal{M}$ admits small colimits, k admits a left adjoint L, and the fact that k preserves small colimits implies that $X \simeq kLX$ for $X \in \mathrm{Model}_{\mathcal{P}}^{\Omega}$. It is then sufficient to verify that $LkM \simeq M$ for $M \in \mathcal{M}$. This is immediate in case (b), and in case (a) follows as k is conservative and $kM \simeq kLkM$. $\square$

If $\mathcal{P}$ is stable, then as $\mathrm{Model}_{\mathcal{P}}^{\Omega}$ is stable, $\mathcal{P}$ is additive. In this case it is not difficult to see that $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ is a complete and cocomplete abelian category with enough projectives, and that every such category arises this way (Proposition 3.2).

We now move on to methods that allow for the production of unstable examples.

Proposition 2.6. Let $\mathcal{P}$ and $\mathcal{P}'$ be theories, and let $f\colon \mathcal{P}' \to \mathcal{P}$ be an essentially surjective coproduct-preserving functor.

(1) Restriction $f^*\colon \mathrm{Model}_{\mathcal{P}} \to \mathrm{Model}_{\mathcal{P}'}$ is the forgetful functor of a monadic adjunction;
(2) If $\mathcal{P}$ and $\mathcal{P}'$ are loop theories and f preserves tensors by finite wedges of spheres, then restriction $f^*\colon \mathrm{Model}_{\mathcal{P}}^{\Omega} \to \mathrm{Model}_{\mathcal{P}'}^{\Omega}$ is the forgetful functor of a monadic adjunction.

Proof. Observe that both instances of f^* are right adjoints, with left adjoints constructed from $f_!$. Moreover, the assumption that f is essentially surjective implies that each f^* is conservative. By Beck's monadicity theorem [**Lur17a**, Theorem 4.7.3.5], it is sufficient to verify that f^* creates f^*-split geometric realizations. Indeed, split geometric realizations are in particular pointwise geometric realizations, so this follows from the fact that f^* is essentially surjective. $\square$

Proposition 2.7. Let $\mathcal{P}$ be a theory, and let $T\colon \mathrm{Model}_{\mathcal{P}} \to \mathrm{Model}_{\mathcal{P}}$ be a monad which preserves geometric realizations, so that T is the left Kan extension of its restriction t to $\mathcal{P}$. Let $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ be the full subcategory spanned by objects of the form $t(P)$ for $P \in \mathcal{P}$. Then $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Model}_{T\mathcal{P}}$.

Proof. This follows by an application of Theorem 2.1. $\square$

Theorem 2.5. Let $\mathcal{P}$ be a loop theory, and let T be an accessible monad on $\mathrm{Model}_{\mathcal{P}}^{\Omega}$. Let $t = Th$ denote the restriction of T to $\mathcal{P}$, and let $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ be the full subcategory spanned by objects of the form $t(P)$ for $P \in \mathcal{P}$. Write $L\colon \mathrm{Model}_{\mathcal{P}} \to \mathrm{Model}_{\mathcal{P}}^{\Omega}$ for the localization. Then the restricted Yoneda embedding yields an equivalence $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Model}_{T\mathcal{P}}^{\Omega}$ if and only if the canonical map $Lt_!X \to TX$ is an equivalence for $X \in \mathrm{Model}_{\mathcal{P}}^{\Omega}$. In particular, this holds if T preserves geometric realizations.

Proof. Observe that there is a factorization of forgetful functors

$$\mathcal{A}\mathrm{lg}_T \xrightarrow{h} \mathrm{Model}_{T\mathcal{P}}^{\Omega} \xrightarrow{t^*} \mathrm{Model}_{\mathcal{P}}^{\Omega} .$$

By Proposition 2.6, both $\mathcal{A}\mathrm{lg}_T \to \mathrm{Model}_{\mathcal{P}}^{\Omega}$ and $\mathrm{Model}_{T\mathcal{P}}^{\Omega} \to \mathrm{Model}_{\mathcal{P}}^{\Omega}$ are the forgetful functors of monadic adjunctions, with associated monads T and $Lt_!$. The above factorization gives rise to a map $Lt_! \to T$ of monads which is an equivalence if and only if $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Model}_{T\mathcal{P}}^{\Omega}$. $\square$

Remark 2.5. We have used the language of monads as it makes the relevant applications more apparent, however this has the downsides of both relying on more technology than is necessary and obscuring some of the underlying logic. This could be avoided by staying on the one side of Beck's theorem; for example, Theorem 2.5 amounts the following statement, which can be proved directly.

Fix a loop theory $\mathcal{P}$, presentable category $\mathcal{D}$, and conservative right adjoint $U\colon \mathcal{D} \to \mathrm{Model}_{\mathcal{P}}^{\Omega}$ which preserves U-split geometric realizations. Write T for the left adjoint, $t = Th$ for the restriction of T to $\mathcal{P}$, and $T\mathcal{P} \subset \mathcal{D}$ for the full subcategory spanned by objects of the form $t(P)$ for $P \in \mathcal{P}$. Then the restricted Yoneda embedding $h\colon \mathcal{D} \to \mathrm{Model}_{T\mathcal{P}}^{\Omega}$ is an equivalence if and only if the diagram

$$\begin{tikzcd} \mathrm{Model}_{\mathcal{P}}^{\Omega} \arrow[r, "T"] \arrow[d, "j"] & \mathcal{D} \arrow[r, "U"] & \mathrm{Model}_{\mathcal{P}}^{\Omega} \\ \mathrm{Model}_{\mathcal{P}} \arrow[r, "t_!"] & \mathrm{Model}_{T\mathcal{P}} \arrow[r, "t^*"] & \mathrm{Model}_{\mathcal{P}} \arrow[u, "L"'] \end{tikzcd}$$

canonically commutes, i.e. the natural transformation $Lt^*t_!j \to UT$ is an equivalence.

This simplifies further if U preserves all geometric realizations. $\triangleleft$

In the situation of Theorem 2.5, we would like to identify the algebraic category $\mathrm{Model}_{T\mathcal{P}}^{\heartsuit}$. To that end, we have the following.

Proposition 2.8. Let $\mathcal{P}$ be a theory, and fix a monad on $\mathrm{Model}_{\mathcal{P}}$ which preserves geometric realizations, and so has underlying functor of the form $t_!$ for some $t\colon \mathcal{P} \to \mathrm{Model}_{\mathcal{P}}$. Let $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_{t_!}$ be the full subcategory spanned by objects of the form $t(P)$ for $P \in \mathcal{P}$, so that $\mathcal{A}\mathrm{lg}_{t_!} \simeq \mathrm{Model}_{T\mathcal{P}}$. Then

(1) $t^*\colon \mathrm{Model}_{T\mathcal{P}}^{\heartsuit} \to \mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ is the forgetful functor of a monadic adjunction; write $\mathbb{T}$ for the associated monad on $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$.

(2) The monad $\mathbb{T}$ is determined by natural isomorphisms $\mathbb{T}(\pi_0 X) \simeq \pi_0 t_! X$ of T-algebras for $X \in \mathrm{Model}_{\mathcal{P}}$;

(3) If $\mathcal{P}$ is a loop theory and $t_!$ is obtained from a monad T on $\mathrm{Model}_{\mathcal{P}}^{\Omega}$, then $\mathbb{T}$ can instead be described in terms of T in the following manner:
   (a) $\mathbb{T}$ preserves reflexive coequalizers;
   (b) There are natural maps $\mathbb{T}(\pi_0 X) \to \pi_0 T X$ for $X \in \mathrm{Model}_{\mathcal{P}}^{\Omega}$ which are isomorphisms when $X = h(P)$ for some $P \in \mathcal{P}$;
   (c) The diagrams

$$\begin{tikzcd} \mathbb{T}\mathbb{T}(\pi_0 X) \arrow[r] \arrow[d, "\mu"] & \mathbb{T}(\pi_0 TX) \arrow[r] & \pi_0 TTX \arrow[d, "\mu"] \\ \mathbb{T}(\pi_0 X) \arrow[rr] & & \pi_0 TX \end{tikzcd} \qquad \begin{tikzcd} \pi_0 X \arrow[r, "\eta"] \arrow[dr, "\eta"'] & \mathbb{T}(\pi_0 X) \arrow[d] \\ & \pi_0 TX \end{tikzcd}$$

commute.

Proof. (1) This follows immediately from the crude monadicity theorem.

(2) First observe that $\mathcal{M}\mathrm{odel}^{\heartsuit}_{T\mathcal{P}}$ can be identified as the category of discrete $t_!$-algebras. In particular, if X is a $t_!$-algebra, then $\pi_0 X$ is a $t_!$-algebra. As a consequence, for $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ the map $X \to t_! X \to \pi_0 t_! X$ extends uniquely to a map $\mathbb{T}(\pi_0 X) \to \pi_0 t_! X$ of $t_!$-algebras, which is evidently an isomorphism when $X = h(P)$ with $P \in \mathcal{P}$. As this is a natural transformation of functors which preserve geometric realizations, as computed in $\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathcal{P}}$, it is a natural isomorphism, verifying (2).

(3) As there are maps $t_! X \to TX$ for $X \in \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$, we can take as our natural transformation the map $\mathbb{T}(\pi_0 X) \simeq \pi_0 t_! X \to \pi_0 TX$; this has the indicated properties as $t(P) = Th(P)$ by assumption, and these evidently determine $\mathbb{T}$. $\square$

In part (3) of Proposition 2.8, the assumption that $t_!$ is obtained from a monad T on $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ is, by Proposition 2.6, equivalent to the assumption that $t(P) \in \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ for $P \in \mathcal{P}$. In Section 2.5, we will consider situations where this can fail; examples where $\mathcal{A}\mathrm{lg}_T \simeq \mathcal{M}\mathrm{odel}^{\Omega}_{T\mathcal{P}}$ even when T does not preserve geometric realizations; and examples where the hypotheses of Theorem 2.4 are not satisfied yet nonetheless $\mathcal{M} \simeq \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$.

2.3. Stable loop theories

This section concerns those loop theories $\mathcal{P}$ that give rise to stable categories. In the stable setting, it is natural to consider the category $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ of spectrum-valued models, and corresponding full subcategory $\mathrm{L}\mathcal{M}\mathrm{od}^{\Omega}_{\mathcal{P}} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ of spectrum-valued models that preserve loops. These categories behave somewhat differently from their unstable versions; the most important aspect is that the inclusion $\mathrm{L}\mathcal{M}\mathrm{od}^{\Omega}_{\mathcal{P}} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ has an explicitly describable left adjoint, which we give in Theorem 2.6.

To illustrate what can be done in this context, we construct in Subsection 2.3.2 a spectral sequence for computing with suitable functors into $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$, and verify in Subsection 2.3.3 that it is multiplicative when one would expect it to be; this relies on the material of Section 2.6. In Subsection 2.3.4, we give a spectral sequence for computing mapping spectra; this can be seen in part as a warmup for the more involved obstruction theory available in unstable contexts given in Subsection 2.4.3, although it is not quite a special case of the latter.

2.3.1. Additive and stable theories. Fix a theory $\mathcal{P}$, and suppose moreover that $\mathcal{P}$ is additive. In this case, it turns out that $\mathrm{Model}_{\mathcal{P}}$ is close to being stable; specifically, it is prestable in the sense of [**Lur18**, Appendix C]. The properties we need are summarized below.

Write $\mathcal{S}\mathrm{p}$ for the category of a spectra, and write $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ for the category of $\mathcal{S}\mathrm{p}$-valued models of $\mathcal{P}$ and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}}$ for the category of $\mathcal{S}\mathrm{p}_{\geq 0}$-valued models of $\mathcal{P}$, where $\mathcal{S}\mathrm{p}_{\geq 0}$ is the category of connective spectra; these are all presentable.

Lemma 2.9. Let $\mathcal{P}$ be an additive loop theory. Then

(1) Postcomposition with Ω^∞ yields an equivalence $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}$;
(2) The embedding $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ realizes the category $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ as the category of spectrum objects of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

Proof. If $\mathcal{P}$ is finitary, then one may appeal directly to [**Lur18**, Remark C.1.5.9]; in general, one may appeal to the same by use of the embedding $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \subset \mathrm{Psh}^{\Pi_\omega}(\mathcal{P})$. More directly, using the description of colimits in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ given by Lemma 2.3, it is seen that $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is additive, from which it follows, as in the proof of [**Lur18**, Proposition C.1.5.7], that $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}}$; the second claim follows in turn as $\Omega\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is computed pointwise. $\square$

We may at times abuse notation by identifying $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ with $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}}$ as a subcategory of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$.

Fix now a stable loop theory $\mathcal{P}$; in particular $\mathcal{P}$ is additive. Write $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ for the full subcategory of objects X such that $X_\Sigma \simeq \Omega X$; this is distinct from the image of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ under $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$.

Theorem 2.6. The inclusion $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ is the inclusion of a reflective subcategory, and the associated localization L on $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ is given by

$$LX = \operatorname*{colim}_{n\to\infty} \Sigma^{-n} X_{\Sigma^{-n}}.$$

Proof. As both $X \mapsto \Sigma^{-1}X$ and $X \mapsto X_{\Sigma^{-1}}$ are automorphisms of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, for $X \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ and $Y \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$ there are equivalences

$$\mathrm{Map}_{\mathcal{P}}(X,Y) \simeq \mathrm{Map}_{\mathcal{P}}(\Sigma^{-1}X_{\Sigma^{-1}}, \Sigma^{-1}Y_{\Sigma^{-1}}) \simeq \mathrm{Map}_{\mathcal{P}}(\Sigma^{-1}X_{\Sigma^{-1}}, Y),$$

with composite given by restriction along $\Sigma^{-1}X_{\Sigma^{-1}} \to X$. Thus if we define $L'X = \operatorname{colim}_{n\to\infty} \Sigma^{-n}X_{\Sigma^{-n}}$, then $\mathrm{Map}_{\mathcal{P}}(X,Y) \simeq \mathrm{Map}_{\mathcal{P}}(L'X,Y)$, and to show $LX \simeq L'X$ must only verify that $L'X \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$. As $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ is stable, finite limits commute past arbitrary colimits. Thus we may compute

$$\Omega L'X \simeq \Omega \operatorname*{colim}_{n\to\infty} \Sigma^{-n} X_{\Sigma^{-n}} \simeq \operatorname*{colim}_{n\to\infty} \Sigma^{-n-1} X_{\Sigma^{-n}} \simeq (\operatorname*{colim}_{n\to\infty} \Sigma^{-n} X_{\Sigma^{-n}})_\Sigma \simeq (L'X)_\Sigma,$$

showing $L'X \in \mathrm{LMod}_{\mathcal{P}}^{\Omega}$. □

Remark 2.6. The inclusion $\mathrm{LMod}_{\mathcal{P}}^{\Omega} \subset \mathrm{LMod}_{\mathcal{P}}$ is also the inclusion of a coreflective category, with colocalization R on $\mathrm{LMod}_{\mathcal{P}}$ given by

$$RX = \lim_{n\to\infty} \Sigma^{-n} X_{\Sigma^{-n}}.$$

We will not make use of this. ◁

Corollary 2.2. When $\mathcal{P}$ is stable,

1. If $X \in \mathrm{Model}_{\mathcal{P}}^{\Omega} \subset \mathrm{LMod}_{\mathcal{P}}^{\mathrm{cn}} \subset \mathrm{LMod}_{\mathcal{P}}$, then the tower of Theorem 2.6 producing LX is exactly the Whitehead tower of LX;
2. Postcomposition with Ω^∞ yields an equivalence $\mathrm{LMod}_{\mathcal{P}}^{\Omega} \simeq \mathrm{Model}_{\mathcal{P}}^{\Omega}$;
3. The full subcategory $\mathrm{LMod}_{\mathcal{P}}^{\Omega} \subset \mathrm{LMod}_{\mathcal{P}}$ is closed under all small limits and colimits;
4. The composite $\mathrm{Model}_{\mathcal{P}} \simeq \mathrm{LMod}_{\mathcal{P}} \to \mathrm{LMod}_{\mathcal{P}}^{\Omega} \simeq \mathrm{Model}_{\mathcal{P}}^{\Omega}$ is left adjoint to the inclusion $\mathrm{Model}_{\mathcal{P}}^{\Omega} \subset \mathrm{Model}_{\mathcal{P}}$.

Proof. (1) This is clear, as the map $X \to \Sigma^{-1} X_{\Sigma^{-1}}$ is an equivalence on (-1)-connected covers.

(2) Under the identification $\mathrm{Model}_{\mathcal{P}} \simeq \mathrm{LMod}_{\mathcal{P}}^{\mathrm{cn}}$, postcomposition with Ω^∞ is identified with $\tau_{\geq 0}$. Observe that if $X \in \mathrm{Model}_{\mathcal{P}}^{\Omega}$ then $(LX)_{\geq 0} \simeq X$ by (1), and thus $\tau_{\geq 0}$ is essentially surjective. To see that it is fully faithful, fix $X, Y \in \mathrm{LMod}_{\mathcal{P}}^{\Omega}$ and compute

$$\mathrm{Map}_{\mathcal{P}}(X,Y) \simeq \mathrm{Map}_{\mathcal{P}}(L\tau_{\geq 0}X, Y) \simeq \mathrm{Map}_{\mathcal{P}}(\tau_{\geq 0}X, Y) \simeq \mathrm{Map}_{\mathcal{P}}(X_{\geq 0}, Y_{\geq 0}).$$

(3) This is clear.

(4) Observe that if $X \in \mathrm{Model}_{\mathcal{P}}$ and $Y \in \mathrm{Model}_{\mathcal{P}}^{\Omega}$, then

$$\mathrm{Map}_{\mathcal{P}}(X,Y) \simeq \mathrm{Map}_{\mathcal{P}}(X, (LY)_{\geq 0}) \simeq \mathrm{Map}_{\mathcal{P}}(X, LY) \simeq \mathrm{Map}_{\mathcal{P}}(LX, LY),$$

so the claim follows from (2). □

Warning 2.1. Part (4) of Corollary 2.2 does not combine with Theorem 2.6 to give an explicit description of the localization $L\colon \mathrm{Model}_{\mathcal{P}} \to \mathrm{Model}_{\mathcal{P}}^{\Omega}$ in general, but it does when $\mathcal{P}$ is finitary. Here the issue is that in general $\Omega^\infty\colon \mathrm{LMod}_{\mathcal{P}} \to \mathrm{Model}_{\mathcal{P}}$ need not preserve filtered colimits. ◁

Any fiber sequence $X \to Y \to Z$ in $\mathrm{Model}_{\mathcal{P}}$ with second map a π_0-surjection remains a fiber sequence in $\mathrm{LMod}_{\mathcal{P}}$. In particular, Theorem 2.3 gives such a fiber sequence $BX_\Sigma \to X \to \tau^*\tau_! X$ in $\mathrm{Model}_{\mathcal{P}}$, yielding the following.

LEMMA 2.10. For $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, there is a fiber sequence

$$\Sigma X_\Sigma \to X \to \tau^* \tau_! X$$

in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$. □

2.3.2. Left-derived functor spectral sequences. Throughout this subsection, we fix a stable loop theory $\mathcal{P}$ and arbitrary loop theory $\mathcal{P}'$.

Let $f\colon \mathcal{P}' \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ be a functor, and extend this to $F\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ by left Kan extension. The composite $\pi_0 \circ f\colon \mathcal{P}' \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$ factors through $\mathrm{h}\mathcal{P}$, and we denote the resulting functor as $\overline{f}\colon \mathrm{h}\mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$. Again by left Kan extension we obtain $\overline{F}\colon \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}'}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$. Recall from Subsection 2.1.2 our discussion of the total left-derived functor $\mathbb{L}\overline{F}$ defined as $\mathbb{L}\overline{F} = \overline{f}_!\colon \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$.

PROPOSITION 2.9. The diagram

$$\begin{tikzcd} \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \arrow[r, "f_!"] \arrow[d, "\tau_!"] & \mathcal{M}\mathrm{odel}_{\mathcal{P}} \arrow[d, "\tau_!"] \\ \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}'} \arrow[r, "\mathbb{L}\overline{F}"] & \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}} \end{tikzcd}$$

commutes.

PROOF. As all functors involved preserve geometric realizations, it suffices to check that the diagram commutes upon restriction to $\mathcal{P}'$. This itself follows from Corollary 2.1 together with the assumption that $f(P') \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ for $P' \in \mathcal{P}'$. □

THEOREM 2.7. Fix notation as above, and fix $R \in \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\Omega}$. Then the spectral sequence in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ associated to the tower

$$FR \simeq \operatorname*{colim}_{n\to\infty} \Sigma^{-n}(f_! R)_{\Sigma^{-n}}$$

guaranteed by Theorem 2.6 is of signature

$$E^1_{p,q} = (\mathbb{L}_{p+q}\overline{F}\pi_0 R)[-p] \Rightarrow (\pi_0 FR)[q], \qquad d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q-1}.$$

This spectral sequence converges, for instance, if π_0 preserves filtered colimits or if each $\mathbb{L}\overline{F}\pi_0 R$ is truncated.

PROOF. By Lemma 2.10 and Proposition 2.9, the tower $FR \simeq \mathrm{colim}_{n\to\infty} \Sigma^{-n}(f_! R)_{\Sigma^{-n}}$ has layers described by cofiber sequences

$$\Sigma^{-(n-1)}(f_! R)_{\Sigma^{-(n-1)}} \to \Sigma^{-n}(f_! R)_{\Sigma^{-n}} \to \tau^* \Sigma^{-n}(\mathbb{L}\overline{F}\pi_0 R)[-n].$$

This gives rise to the indicated spectral sequence in the usual way; we will review the construction and convergence in Subsection 2.6.1. □

The method of constructing spectral sequences by analyzing the tower obtained from Theorem 2.6 is more general than just that given in Theorem 2.7. Roughly, given $M \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, to obtain a tool for computing $\pi_* M$ one wants to find some $M' \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ with $LM' = M$ such that $\tau_! M'$ is something computable. In the preceding theorem, we had $M = FX$, and took $M' = f_! X$; another simple case is the following.

Example 2.1. Observe that each of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$, $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, and $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$ are tensored over $\mathcal{S}\mathrm{p}_{\geq 0}$ by additivity. Denote the resulting tensors by $\otimes_!$, $\otimes$, and $\overline{\otimes}^{\mathbb{L}}$. These are all compatible, in that if $X \in \mathcal{S}\mathrm{p}_{\geq 0}$ and $M \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, then

$$\tau_!(X \otimes_! M) = X \overline{\otimes}^{\mathbb{L}} \tau_! M, \qquad L(X \otimes_! M) = X \otimes LM.$$

For $X \in \mathcal{S}\mathrm{p}_{\geq 0}$ and $\Lambda \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, write

$$\pi_*(X \overline{\otimes}^{\mathbb{L}} \Lambda) = H_*(X; \Lambda);$$

this is a form of ordinary homology. For $M \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, we obtain an Atiyah-Hirzebruch-type spectral sequence

$$E^1_{p,q} = H_{p+q}(X; \pi_0 M)[-p] \Rightarrow \pi_q(X \otimes M), \qquad d^r \colon E^r_{p,q} \to E^r_{p-r,q-1},$$

by analyzing the tower $X \otimes M = \operatorname{colim}_{n \to \infty} \Sigma^{-n}(X \otimes_! M)_{\Sigma^{-n}}$. ◁

2.3.3. Monoidal matters. This subsection relies on the material of Section 2.6.

We would like to introduce monoidal properties of the constructions discussed in Subsection 2.3.1 and Subsection 2.3.2. Our primary reason for doing so is to introduce pairings into the spectral sequence of Theorem 2.7. For the sake of completeness, we will work briefly in a more general setting than is necessary for just the production of these pairings, and for this generality we require the theory of ∞-operads as developed in [**Lur17a**]. However, the cases of the general theory necessary for our primary application, Theorem 2.8, are just as easily performed by hand, completely bypassing the theory of ∞-operads.

Fix a single-colored ∞-operad $\mathcal{O}$ in the sense of [**Lur17a**]. We will implicitly use the fact that every symmetric monoidal category canonically inherits the structure of an $\mathcal{O}$-monoidal category. Say that an $\mathcal{O}$-monoidal structure on a category $\mathcal{D}$ *respects* some class of colimits in $\mathcal{D}$ if for every $n \geq 0$ and $f \in \mathcal{O}(n)$, the tensor product $\otimes_f$ preserves such colimits in each variable. An $\mathcal{O}$-monoidal category $\mathcal{D}$ is said to be $\mathcal{O}$*-monoidally cocomplete* if it admits small colimits and its $\mathcal{O}$-monoidal structure respects these. In [**Lur17a**, Section 2.2.6], it is shown that if $\mathcal{C}$ is a small $\mathcal{O}$-monoidal category and $\mathcal{D}$ is an $\mathcal{O}$-monoidally cocomplete category, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits the structure of an $\mathcal{O}$-monoidally cocomplete category under Day convolution, informally described as follows: for $n \geq 0$, $f \in \mathcal{O}(n)$, and $F_1, \ldots, F_n \colon \mathcal{C} \to \mathcal{D}$, the tensor product $\otimes_f(F_1, \ldots, F_n)$ is the left Kan extension of

$\otimes_f \circ (F_1 \times \cdots \times F_n) \colon \mathcal{C}^{\times n} \to \mathcal{D}^{\times n} \to \mathcal{D}$ along $\otimes_f \colon \mathcal{C}^{\times n} \to \mathcal{C}$. Of interest is the case where $\mathcal{C}$ is the poset $(\mathbb{Z}, <)$ with symmetric monoidal structure given by addition, where for towers $X_1, \ldots, X_n$ in $\mathcal{D}$ we identify

$$\otimes_f(X_1, \ldots, X_n)(p) = \operatorname*{colim}_{p_1+\cdots+p_n \leq p} \otimes_f(X_1(p_1), \ldots, X_n(p_n)).$$

Definition 2.4. A loop theory $\mathcal{P}$ is an $\mathcal{O}$-*monoidal loop theory* if it is equipped with an $\mathcal{O}$-monoidal structure compatible with coproducts and tensors by finite wedges of spheres. $\triangleleft$

Fix an $\mathcal{O}$-monoidal loop theory $\mathcal{P}$. We obtain by Day convolution, following [**Lur17a**, Proposition 4.8.1.10], $\mathcal{O}$-monoidal categories, all compatible with colimits, and strong $\mathcal{O}$-monoidal functors, fitting into the diagram

$$\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \longleftarrow \mathcal{M}\mathrm{odel}_{\mathcal{P}} \longrightarrow \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} .$$

When $\mathcal{P}$ is in addition stable, we similarly obtain

$$\begin{tikzcd} \mathcal{M}\mathrm{odel}_{\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \arrow[d, "\simeq"] \\ \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \arrow[r] & \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \end{tikzcd} .$$

There is an evident notion of an $\mathcal{O}$-monoidal theory, and if $\mathcal{P}$ is such then we obtain the same diagrams, only with $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$ omitted. The only thing we have to say in this level of generality is the following.

Proposition 2.10. Suppose that $\mathcal{P}$ is an $\mathcal{O}$-monoidal stable loop theory. Then the functor $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{Fun}(\mathbb{Z}, \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}})$ sending X to the tower

$$\cdots \to \Sigma X_{\Sigma} \to X \to \Sigma^{-1} X_{\Sigma^{-1}} \to \cdots$$

is canonically lax $\mathcal{O}$-monoidal.

Proof. This functor preserves colimits, so by the universal property of Day convolution it is sufficient to verify that it is canonically lax $\mathcal{O}$-monoidal upon restriction to $\mathcal{P}$. By Corollary 2.2, its restriction to $\mathcal{P}$ is equivalent to the composite

$$\mathcal{P} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \xrightarrow{W} \mathrm{Fun}(\mathbb{Z}, \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}),$$

where W is the functor sending an object to its Whitehead tower. We conclude by applying Proposition 2.26. $\square$

We restrict now to the case where $\mathcal{O}$ is the nonunital $\mathbb{A}_2$-operad, i.e. where our monoidal structures consist merely of a single pairing subject to no further coherence conditions. This is both the most general and the simplest situation: in the context of pairings of spectral sequences, additional properties such as associativity and commutativity can be verified at the level of homotopy groups, so we need not be concern ourselves with the coherence problems they present. Fix nonunital $\mathbb{A}_2$-monoidal loop theories $\mathcal{P}$ and $\mathcal{P}'$, and suppose that $\mathcal{P}$ is stable. Write the associated pairings on $\mathrm{L}\mathcal{M}\mathrm{od}^{\Omega}_{\mathcal{P}}$ and $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}'}$ as $\otimes$, and the associated pairings on $\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}}$ and $\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}'}$ as $\overline{\otimes}$.

Fix a functor $F\colon \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ which preserves geometric realizations; from here we will use notation as in Subsection 2.3.2. Suppose that F is lax monoidal; equivalently, that we have chosen a natural transformation $f(P') \otimes f(Q') \to f(P' \otimes Q')$. This gives rise to lax monoidal structures on $f_!$, $\overline{F}$, and $\mathbb{L}\overline{F}$.

By Theorem 2.2 and the classic Dold-Kan correspondence [**DP61**, Section 3], $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$ can be modeled as $\mathrm{s}\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}} \simeq \mathrm{Ch}^+(\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}})$. Following Proposition 2.4, the pairing on $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$ induced by that on $\mathrm{h}\mathcal{P}$ can be modeled by the levelwise pairing on $\mathrm{s}(\mathrm{h}\mathcal{P}) \subset \mathrm{s}\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}}$, and by the Eilenberg-Zilber theorem this is modeled by the standard pairing on $\mathrm{Ch}^+(\mathrm{h}\mathcal{P}) \subset \mathrm{Ch}^+(\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}})$. To be precise, we choose the pairing on $\mathrm{Ch}^+(\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}})$ given by $(C' \mathbin{\overline{\otimes}} C'')_p = \bigoplus_{p'+p''} C'_{p'} \mathbin{\overline{\otimes}} C''_{p''}$, with differential $d(x' \otimes x'') = d(x') \otimes x'' + (-1)^{|x'|} x' \otimes d(x'')$. With this choice, a pairing $C' \mathbin{\overline{\otimes}} C'' \to C$ of chain complexes gives Künneth maps $H_{q'}C' \mathbin{\overline{\otimes}} H_{q''}C'' \to H_{q'+q''}C$. From this, for $R, S \in \mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{P}'}$ we obtain pairings $\mathbb{L}_p\overline{F}(R) \mathbin{\overline{\otimes}} \mathbb{L}_q\overline{F}(S) \to \mathbb{L}_{p+q}\overline{F}(R \mathbin{\overline{\otimes}} S)$.

Theorem 2.8. Fix notation as above, and given $X \in \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}'}$, write $E(X)$ for the spectral sequence of Theorem 2.7 computing $\pi_* F(X)$. Then a pairing $X' \otimes X'' \to X$ in $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}'}$ gives rise to a pairing $E(X') \mathbin{\overline{\otimes}} E(X'') \to E(X)$ of spectral sequences, i.e. maps

$$\smile\colon E^r_{p',q'}(X') \mathbin{\overline{\otimes}} E^r_{p'',q''}(X'') \to E^r_{p'+p'',q'+q''}(X)$$

satisfying

$$d^r(x' \smile x'') = d^r(x') \smile x'' + (-1)^{q'} x \smile d^r(x''),$$

with the pairing on E^{r+1} induced by that on E^r. When $r = 1$, this is the algebraic pairing on $\mathbb{L}_*\overline{F}$ twisted by $(-1)^{q'p''}$.

Proof. For the construction of the pairings, combine Proposition 2.10 and Theorem 2.14. By this construction and the identification of the E^1 pages of these spectral sequences, the diagram

$$\begin{tikzcd}
E^1_{p',q'}(X') \mathbin{\overline{\otimes}} E^1_{p'',q''}(X'') \arrow[r, "="] \arrow[dd, "\smile"] & (\mathbb{L}_{p'+q'}\overline{F}\pi_0 X')[-p'] \mathbin{\overline{\otimes}} (\mathbb{L}_{p''+q''}\overline{F}\pi_0 X'')[-p''] \arrow[d] \\
& (\mathbb{L}_{p'+q'+p''+q''}\overline{F}\pi_0 X)[-p'-p''] \arrow[d, "\simeq"] \\
E^1_{p'+q',p''+q''}(X'') \arrow[r, "="] & (\mathbb{L}_{p'+p''+q'+q''}\overline{F}\pi_0 X)[-(p'+p'')]
\end{tikzcd}$$

commutes, where the top right vertical map is the algebraic pairing, and the bottom right vertical map induces a sign of $(-1)^{q'p''}$. $\square$

The pairings produced by Theorem 2.8 behave well with respect to the pairing $F(X')\otimes F(X'')\to F(X)$; see our comments at the end of Subsection 2.6.3.

2.3.4. Universal coefficient spectral sequences. Fix a stable loop theory $\mathcal{P}$. For $X,Y\in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, there is a mapping spectrum $\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X,Y)$ with $\Omega^{\infty-n}\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X,Y)=\mathrm{Map}_{\mathcal{P}}(X,\Sigma^n Y)$.

THEOREM 2.9. Fix $X,Y\in\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$. Then the spectral sequence associated to the Postnikov decomposition

$$\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X,Y)\simeq\lim_{n\to\infty}\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X_{\geq 0},\tau_{\leq n}Y)$$

is of signature

$$E_1^{p,q}=\mathrm{Ext}_{\mathrm{h}\mathcal{P}}^{p+q}(\pi_0 X;\pi_0 Y[p])\Rightarrow\pi_{-q}\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X,Y),\qquad d_r^{p,q}\colon E_r^{p,q}\to E_r^{p+r,q+1}.$$

PROOF. The tower $\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X,Y)\simeq\lim_{n\to\infty}\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X_{\geq 0},\tau_{\leq n}Y)$ has layers described by fiber sequences

$$\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X_{\geq 0},\Sigma^p\pi_0 Y[p])\to\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X_{\geq 0},Y_{\leq p})\to\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X_{\geq 0},Y_{\leq p-1}).$$

By Lemma 2.10, there is an equivalence

$$\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X_{\geq 0},\Sigma^p\pi_0 Y[p])\simeq\mathcal{E}\mathrm{xt}_{\mathrm{h}\mathcal{P}}(\pi_0 X,\Sigma^p\pi_0 Y[p]),$$

so that

$$\pi_{-q}\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X_{\geq 0},\Sigma^p\pi_0 Y[p])=\mathrm{Ext}_{\mathrm{h}\mathcal{P}}^{p+q}(\pi_0 X,\pi_0 Y[p]).$$

The theorem now follows from the usual construction of the spectral sequence of a tower, which we will review in Section 2.6. $\square$

We end with a remark concerning the introduction of extra structure into Theorem 2.9. Suppose that $\mathcal{P}$ is a nonunital $\mathbb{A}_2$-monoidal stable loop theory, and write the resulting pairing on $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ by $\otimes_!$. Then $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ is closed monoidal, in that for $X,Y\in\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ there are objects $F_l(X,Y),F_r(X,Y)\in\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ with

$$\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X,F_l(Y,Z))\simeq\mathcal{E}\mathrm{xt}_{\mathcal{P}}(Y\otimes_! X,Z),\qquad\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X,F_r(Y,Z))\simeq\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X\otimes_! Y,Z).$$

Consider just F_r. Here $F_r(X,Y)(P)=\mathcal{E}\mathrm{xt}_{\mathcal{P}}(h(P)\otimes_! X,Y)$; in particular, if $\otimes_!$ admits a left unit $I\in\mathcal{P}$, then $F_r(X,Y)(I)\simeq\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X,Y)$. The same remarks hold for $\mathrm{h}\mathcal{P}$, so that, at least if $\otimes_!$ admits a left unit, for $X\in\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$ and $M\in\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ we can view $\pi_{-q}F_r(X_{\geq 0},M)\in\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$ as an enrichment of $\mathrm{Ext}_{\mathrm{h}\mathcal{P}}^q(\pi_0 X,M)$, and in this manner obtain an enriched form of Theorem 2.9.

2.4. Postnikov decompositions

Fix a loop theory $\mathcal{P}$. This section considers the study of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ via Postnikov decompositions in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$. We begin in Subsection 2.4.1 with a brief review of the general theory of Postnikov decompositions available in any ∞-topos, then specialize in Subsection 2.4.2 to the case of Postnikov towers $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$, which can be computed in the ambient ∞-topos $\mathrm{Psh}(\mathcal{P})$; see also [**Pst17**] for a treatment of these topics.

Given these generalities, the construction of an obstruction theory for mapping spaces in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is essentially immediate, and obtained in Subsection 2.4.3. Finally, Subsection 2.4.4 contains a verification that the Blanc-Dwyer-Goerss obstruction theory for realizations holds in our setting.

2.4.1. Eilenberg-MacLane objects and Postnikov towers. Fix a Grothendieck ∞-topos $\mathcal{X}$. Up to size issues, which for our purposes can be safely ignored, $\mathcal{X}$ admits an *object classifier* Ω; see [**Lur17b**, Theorem 6.1.6.8]. In other words, there is a universal map $\Omega^* \to \Omega$ in $\mathcal{X}$, pulling back along which induces an equivalence

$$\mathrm{Map}_{\mathcal{X}}(X, \Omega) \simeq (\mathcal{X}/X)^{\simeq} \simeq \coprod_{f \in \mathcal{X}/X} B\mathrm{Aut}_{\mathcal{X}/X}(f)$$

for any $X \in \mathcal{X}$. For $n \geq 1$, there is a subobject $\mathcal{EM}_n \subset \Omega$ classifying abelian Eilenberg-MacLane objects concentrated in degree n, with associated universal map $\mathcal{EM}_n^* \to \mathcal{EM}_n$; write $\mathcal{EM}_1'$ for the object classifying arbitrary Eilenberg-MacLane objects concentrated in degree 1. There are also objects $\mathcal{AB}$ and $\mathcal{GP}$ classifying discrete abelian groups and discrete groups in $\mathcal{X}$ respectively, and following [**Lur17b**, Proposition 7.2.2.12], there are equivalences $\pi_n \colon \mathcal{EM}_n^* \to \mathcal{AB}$ and $\pi_1 \colon \mathcal{EM}_1'^* \to \mathcal{GP}$ with inverses $B^n \colon \mathcal{AB} \to \mathcal{EM}_n^*$ and $B \colon \mathcal{GP} \to \mathcal{EM}_1'^*$. If $X \in \mathcal{X}$, then $\mathrm{Map}_{\mathcal{X}}(X, \mathcal{EM}_n)_{\leq 1} \simeq \mathrm{Map}_{\mathcal{X}}(X, \mathcal{AB})$, allowing us to construct $\pi_n \colon \mathcal{EM}_n \to \mathcal{AB}$ splitting $B^n \colon \mathcal{AB} \to \mathcal{EM}_n$. We can summarize some of the relations between these as follows.

Proposition 2.11. There are Cartesian squares

$$\begin{tikzcd} \mathcal{EM}_n \arrow[r, "\pi_n"] \arrow[d, "\pi_n"] & \mathcal{AB} \arrow[d, "B^{n+1}"] \\ \mathcal{AB} \arrow[r, "B^{n+1}"] & \mathcal{EM}_{n+1} \end{tikzcd}$$

for $n \geq 1$. In other words, $\pi_n \colon \mathcal{EM}_n \to \mathcal{AB}$ makes $\mathcal{EM}_n$ into an object of $\mathcal{EM}_{n+1}^*(\mathcal{X}/\mathcal{AB})$, with pointing given by B^n.

Proof. For any $X \in \mathcal{X}$, the Cartesian product of the given square with X is the original square taken with respect to the slice topos $\mathcal{X}/X$, so it is sufficient to verify that it is Cartesian upon taking global sections. Taking global sections and looking at path components corresponding to some $M \in \mathcal{AB}(\mathcal{X})$, it is sufficient to verify that

$$\begin{tikzcd} B\mathrm{Aut}_{\mathcal{X}}(B^n M) \arrow[r, "\pi_n"] \arrow[d, "\pi_n"] & B\mathrm{Aut}_{\mathcal{A}\mathrm{b}(\mathcal{X})}(M) \arrow[d, "B^{n+1}"] \\ B\mathrm{Aut}_{\mathcal{A}\mathrm{b}(\mathcal{X})}(M) \arrow[r, "B^{n+1}"] & B\mathrm{Aut}_{\mathcal{X}}(B^{n+1}M) \end{tikzcd}$$

is Cartesian. The structure of $\mathcal{AB} \simeq \mathcal{EM}_n^* \to \mathcal{EM}_n$ gives a fiber sequence

$$B^n\mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, M) \to B\mathrm{Aut}_{\mathcal{A}\mathrm{b}(\mathcal{X})}(M) \to B\mathrm{Aut}_{\mathcal{X}}(B^n M),$$

and because M is abelian, this is split by π_n. We can thus identify

$$\Omega\,\mathrm{Fib}(\pi_n) \simeq \mathrm{Fib}(B^n) \simeq B^n\mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, M), \qquad \mathrm{Fib}(B^{n+1}) \simeq B^{n+1}\mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, M),$$

and so find that the above square is Cartesian by comparing fibers. $\square$

A map $X \to \mathcal{AB}$ classifies a discrete abelian group in $\mathcal{X}/X$; call such an object an *X-module*. As $\mathcal{AB}$ is 1-truncated, X-modules are equivalent to $X_{\leq 1}$-modules. As moreover $X \to \pi_0 X$ is 1-connected, $\mathrm{Map}_{\mathcal{X}}(\pi_0 X, \mathcal{AB}) \to \mathrm{Map}_{\mathcal{X}}(X, \mathcal{AB})$ is (-1)-truncated, i.e. is an inclusion of a collection of path components; call an X-module *simple* if it is in the image of this map. A theory of Postnikov towers arises from the observation that for all $n \geq 2$, there are Cartesian squares

$$\begin{tikzcd} X_{\leq n} \arrow[r] \arrow[d] & \mathcal{AB} \arrow[d] \\ X_{\leq n-1} \arrow[r] & \mathcal{EM}_n \end{tikzcd},$$

and for $n = 1$ there is an analogous square with $\mathcal{EM}_1$ replaced by $\mathcal{EM}_1'$ and $\mathcal{AB}$ by $\mathcal{GP}$. If when $n = 1$ such a replacement is not necessary, say that X has *abelian homotopy groups*. The top horizontal map of the above square defines an X-module $\Pi_n X$ for $n \geq 2$, or for $n \geq 1$ if X has abelian homotopy groups. If X has abelian homotopy groups and $\Pi_n X$ is simple for $n \geq 1$, say that X is *simple*.

For $\Lambda \in \mathcal{X}_{\leq 1}$ and M a Λ-module, one may form the Eilenberg-MacLane objects $B^{n+1}_\Lambda M$ in $\mathcal{X}/\Lambda$ for all $n \geq 0$. When $n \geq 1$, these fit into Cartesian squares

$$\begin{tikzcd} B^{n+1}_\Lambda M \arrow[r] \arrow[d] & \mathcal{EM}_n \arrow[d, "\pi_n"] \\ \Lambda \arrow[r, "M"] & \mathcal{AB} \end{tikzcd}.$$

Proposition 2.12. Fix $X \in \mathcal{X}$. For $n \geq 2$, there is a natural Cartesian square

$$\begin{tikzcd} X_{\leq n} \arrow[r] \arrow[d] & X_{\leq 1} \arrow[d] \\ X_{\leq n-1} \arrow[r] & B^{n+1}_{X_{\leq 1}}\Pi_n X \end{tikzcd}$$

in $\mathcal{X}$. If X is simple, then for $n \geq 1$ there is a natural Cartesian square

$$\begin{tikzcd} X_{\leq n} \arrow[r] \arrow[d] & \pi_0 X \arrow[d] \\ X_{\leq n-1} \arrow[r] & B^{n+1}_{\pi_0 X}\Pi_n X \end{tikzcd}$$

in $\mathcal{X}$.

Proof. Both cases are handled the same way. Consider the diagram

$$\begin{tikzcd} X_{\leq n} \arrow[r] \arrow[d] & X_{\leq 1} \arrow[r] \arrow[d] & \mathcal{AB} \arrow[d] \\ X_{\leq n-1} \arrow[r, dashed, "k"] \arrow[dr] & B^{n+1}_{X_{\leq 1}}\Pi_n X \arrow[r, "j"] \arrow[d] & \mathcal{EM}_n \arrow[d] \\ & X_{\leq 1} \arrow[r] & \mathcal{AB} \end{tikzcd} .$$

Here, the map $j \circ k$ exists making the upper half of the diagram Cartesian and the bottom half commute, so the individual map k exists as the bottom right square is Cartesian. As the upper half of the diagram is Cartesian, to show that the upper left square is Cartesian it is sufficient to verify that the upper right square is Cartesian. As the bottom right square is Cartesian, it is sufficient to verify that the right half of the diagram is Cartesian, which is clear. $\square$

2.4.2. The Postnikov tower of a model of a theory. Fix a theory $\mathcal{P}$, and set $\mathcal{X} = \mathrm{Psh}(\mathcal{P})$. Observe that $\mathrm{Model}_{\mathcal{P}} \subset \mathcal{X}$ is closed under Postnikov towers.

Proposition 2.13 ([**Pst17**, Lemma 2.64]). Fix $X \in \mathrm{Model}_{\mathcal{P}}$. Then any X-module in $\mathcal{AB}(\mathrm{Model}_{\mathcal{P}}/X) \subset \mathcal{AB}(\mathcal{X}/X)$ is simple. In particular each $\Pi_n X$ is simple, and so X is simple.

Proof. By the equivalence between X-modules and $X_{\leq 1}$-modules, we may suppose that X is 1-truncated, so that everything is taking place within the bicategory $\mathrm{Psh}(\mathcal{P}, \mathcal{G}\mathrm{pd})$. Let $\pi\colon E \to X$ be an X-module. Lemma 2.1 shows that for all $P \in \mathcal{P}$, the covering map $\pi_P\colon E(P) \to X(P)$ has trivial monodromy. The module E is classified by the map $c\colon X \to \mathcal{AB}$ given for $P \in \mathcal{P}$ by the functor $c_P\colon X(P) \to \mathcal{A}\mathrm{b}(\mathrm{Psh}(\mathcal{P}/P, \mathcal{S}\mathrm{et}))^{\simeq}$ defined on objects by $c_P(x)(f\colon Q \to P) = \pi_Q^{-1}(f^* x)$ and on morphisms by monodromy; as a consequence, each c_P factors through $\pi_0 X(P)$. By replacing $\mathcal{P}$ with its homotopy 2-category, and strictifying X to a 2-functor and c to a strict natural transformation, it is seen that this pointwise factorization lifts to factor c through $\pi_0 X$. $\square$

Observe that if $\Lambda \in \mathcal{X}_{\leq 0}$, then $(\mathcal{X}/\Lambda)_{\leq 0} \simeq \mathcal{X}_{\leq 0}/\Lambda$. Thus, given $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, we may form the abuses of notation $\pi_0 X = \tau^*\pi_0 X$, and $\Pi_n X = \tau^*\Pi_n X$ as $\pi_0 X$-modules. Moreover, if $\Lambda \in \mathrm{Psh}(\mathrm{h}\mathcal{P}, \mathcal{S}\mathrm{et})$ and M is a Λ-module, then $\tau^* B^n_\Lambda = B^n_{\tau^*\Lambda}\tau^* M$. This leads to the following.

THEOREM 2.10. For $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ and $n \geq 1$, there is a natural Cartesian square

$$\begin{tikzcd} X_{\leq n} \arrow[r] \arrow[d] & \pi_0 X \arrow[d] \\ X_{\leq n-1} \arrow[r] & \tau^* B^{n+1}_{\pi_0 X}\Pi_n X \end{tikzcd}$$

in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

PROOF. Immediate from Proposition 2.12 and Proposition 2.13. $\square$

2.4.3. An obstruction theory for mapping spaces. Fix a loop theory $\mathcal{P}$, and let $\mathcal{X} = \mathrm{Psh}(\mathcal{P})$. For $Y \in \mathcal{X}$, we may identify $\mathcal{X}/X \simeq \mathrm{Psh}(\mathcal{P}/X)$, where $\mathcal{P}/X$ is the slice category of $\mathcal{P}$ over X. Given a map $f\colon X \to Y$ in $\mathcal{X}$, we may form the objects $\pi_0 f$ and $\Pi_n f$ for $n \geq 1$, considered as objects of $\mathcal{X}/X$. If $X, Y \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, then each $\Pi_n f$ is a simple X-module by Proposition 2.13.

THEOREM 2.11. Fix a π_0-surjection $R \to S$ in $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$. Fix $A, C \in R/\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}/S$, and write $p\colon C \to S$ for the given map. Fix $\phi\colon \pi_0 A \to \pi_0 C$ in $\pi_0 R/\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathcal{P}}/\pi_0 S$, and let $\mathrm{Map}^{\phi}_{R/\mathcal{P}/S}(A, C)$ be the space of lifts of ϕ to a map in $R/\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}/S$. Then the Postnikov tower of p gives rise to a decomposition

$$\mathrm{Map}^{\phi}_{R/\mathcal{P}/S}(A, C) \simeq \lim_{n\to\infty} \mathrm{Map}^{\phi,\leq n}_{R/\mathcal{P}/S}(A, C),$$

where $\mathrm{Map}^{\phi,\leq 0}_{R/\mathcal{P}/S}(A, C) \simeq \{\phi\}$, and for each $n \geq 1$ there is a natural Cartesian square

$$\begin{tikzcd} \mathrm{Map}^{\phi,\leq n}_{R/\mathcal{P}/S}(A, C) \arrow[r] \arrow[d] & \{\phi\} \arrow[d] \\ \mathrm{Map}^{\phi,\leq n-1}_{R/\mathcal{P}/S}(A, C) \arrow[r] & \mathrm{Map}_{\pi_0 R/h\mathcal{P}/\pi_0 S}(\pi_0 A, B^{n+1}_{\pi_0 C}\Pi_n p) \end{tikzcd}.$$

PROOF. Explicitly, this decomposition is obtained from

$$\mathrm{Map}_{R/\mathcal{P}/S}(A, C) \simeq \lim_{n\to\infty} \mathrm{Map}_{R/\mathcal{P}/S}(A, p_{\leq n}),$$

where $p_{\leq n}$ is the n'th Postnikov truncation of C viewed as an object of the slice topos $\mathcal{X}/S$. The layers of this tower fit into Cartesian squares

$$\begin{tikzcd} \mathrm{Map}_{R/\mathcal{P}/S}(A, p_{\leq n}) \arrow[r] \arrow[d] & \mathrm{Map}_{R/\mathcal{P}/S}(A, \pi_0 p) \arrow[d] \\ \mathrm{Map}_{R/\mathcal{P}/S}(A, p_{\leq n-1}) \arrow[r] & \mathrm{Map}_{R/\mathcal{P}/S}(A, B^{n+1}_{\pi_0 p}\Pi_n p) \end{tikzcd},$$

so we claim first that $\mathrm{Map}_{R/\mathcal{P}/S}(A,\pi_0 p) \simeq \mathrm{Hom}_{\pi_0 R/h\mathcal{P}/\pi_0 S}(\pi_0 A,\pi_0 C)$. By resolving A, we reduce to the case where $A = R \amalg P$ for some $q\colon P \to S$. In this case

$$\mathrm{Map}_{R/\mathcal{P}/S}(R\amalg P,\pi_0 p) \simeq \mathrm{Map}_{\mathcal{P}/S}(P,\pi_0 p) \simeq (\pi_0 p)(q\colon P \to S) = \pi_0 F,$$

where F is the fiber of the map $C(P) \to S(P)$ over q. As the composite $R \to C \to S$ is a π_0-surjection, the map $C \to S$ is a π_0-surjection. As $C, S \in \mathrm{Model}_{\mathcal{P}}^{\Omega}$, it follows that $\pi_1 C(P) \to \pi_1 S(P)$ is a surjection at all basepoints. Thus the fiber sequence $F \to C(P) \to S(P)$ remains a fiber sequence on taking π_0; as the fiber of $\pi_0 C(P) \to \pi_0 S(P)$ over q is $\mathrm{Map}_{\pi_0 R/h\mathcal{P}/\pi_0 S}(\pi_0(R\amalg P),\pi_0 C)$, this gives $\mathrm{Map}_{R/\mathcal{P}/S}(A,\pi_0 p) \simeq \mathrm{Hom}_{\pi_0 R/h\mathcal{P}/\pi_0 S}(\pi_0 A,\pi_0 C)$ as claimed. Next, by restricting to path components corresponding to ϕ in the above square, we reduce to identifying the space $\mathrm{Map}_{R/\mathcal{P}/\pi_0 p}(A, B^{n+1}_{\pi_0 p}\Pi_n p)$. This space may be identified as

$$\mathrm{Map}_{R/\mathcal{P}/\pi_0 p}(A, B^{n+1}_{\pi_0 p}\Pi_n p) \simeq \mathrm{Map}_{R/\mathcal{P}/\pi_0 C}(A, B^{n+1}_{\pi_0 C}\Pi_n p) \simeq \mathrm{Map}_{\tau_! R/h\mathcal{P}/\pi_0 C}(\tau_! A, B^{n+1}_{\pi_0 C}\Pi_n p),$$

and we conclude by Corollary 2.1. □

Remark 2.7. The preceding theorem and its proof simplifies upon omitting R and S, whereupon one obtains a decomposition of mapping spaces in $\mathrm{Model}_{\mathcal{P}}$. The somewhat technical nature of the theorem as given is necessary to deal with the following subtlety: if $\mathcal{P}$ is a theory and $X \in \mathrm{Model}_{\mathcal{P}}$, then one might define theories $\mathcal{P}/X$ and $X/\mathcal{P}$ such that $\mathrm{Model}_{\mathcal{P}/X} \simeq \mathrm{Model}_{\mathcal{P}}/X$ and $\mathrm{Model}_{X/\mathcal{P}} \simeq X/\mathrm{Model}_{\mathcal{P}}$; however in general both the maps $\mathrm{h}(\mathcal{P}/X) \to \mathrm{h}\mathcal{P}/\pi_0 X$ and $\mathrm{h}(X/\mathcal{P}) \to \pi_0 X/\mathrm{h}\mathcal{P}$ may fail to be equivalences. ◁

2.4.4. An obstruction theory for realizations. Fix a loop theory $\mathcal{P}$. In the case where $\mathcal{P}$ is pointed and finitary, Pstrągowski [**Pst17**] set up an obstruction theory for realizing an object $\Lambda \in \mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ as $\Lambda = \pi_0 X$ for some $X \in \mathrm{Model}_{\mathcal{P}}^{\Omega}$. In this subsection, we verify that the same obstruction theory exists for a general $\mathcal{P}$; the proof is essentially the same, only with minor modifications necessary to handle the unpointed setting.

We begin with a matter that could have been considered in Subsection 2.4.2. Fix an ∞-topos $\mathcal{X}$; we will soon specialize to $\mathcal{X} = \mathrm{Psh}(\mathcal{P})$. Fix an $(n-1)$-truncated object Y, and set $\pi_0 Y = \Lambda$. Let M be a Λ-module. Every π_0-equivalence $Y \to B^{n+1}_{\Lambda} M$ gives rise, by pulling back along the zero section $\Lambda \to B^{n+1}_{\Lambda} M$, to an n-truncated object X such that $X_{\leq n-1} \simeq Y$ and $\Pi_n X \simeq M$ as Λ-modules; let $\mathcal{M}(Y +_{\Lambda} (M,n)) \subset \mathcal{X}^{\simeq}$ be the space of such X. If we write $\mathrm{Map}^{0\text{-Eq}}$ for spaces of π_0-equivalences, then we obtain a map $\mathrm{Map}^{0\text{-Eq}}_{\mathcal{X}}(Y, B^{n+1}_{\Lambda} M) \to \mathcal{M}(Y +_{\Lambda} (M,n))$. Let also $\mathcal{M}(Y) \subset \mathcal{X}^{\simeq}$ be the space of objects equivalent to Y, and let $\mathrm{Aut}(\Lambda, M)$ be the discrete group of pairs $(\alpha\colon \Lambda \simeq \Lambda, f\colon M \simeq \alpha^* M)$, so that $B\,\mathrm{Aut}(\Lambda, M)$ is equivalent to a path component of $(\mathcal{X}/\mathcal{AB})^{\simeq}$. Then $X \mapsto (X_{\leq n-1}, \Pi_n X)$ determines a map $\mathcal{M}(Y +_{\Lambda} (M,n)) \to \mathcal{M}(Y) \times_{B\,\mathrm{Aut}(\Lambda)} B\,\mathrm{Aut}(\Lambda, M)$. Following [**Pst17**, Theorem 2.71, Remark 3.17], we obtain the following.

Proposition 2.14. The above constructions describe Cartesian squares

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{X}/\Lambda}(Y, B^{n+1}_\Lambda M) \arrow[r] \arrow[d] & \mathrm{Map}^{0\text{-Eq}}_{\mathcal{X}}(Y, B^{n+1}_\Lambda M) \arrow[r] \arrow[d] & \mathcal{M}(Y +_\Lambda (M, n)) \arrow[d] \\
* \arrow[r] & \mathrm{Aut}(\Lambda) \arrow[r] \arrow[d] & \mathcal{M}(Y) \times_{B\mathrm{Aut}(\Lambda)} B\mathrm{Aut}(\Lambda, M) \arrow[d] \\
 & * \arrow[r] & \mathcal{M}(Y) \times B\mathrm{Aut}(\Lambda, M)
\end{tikzcd}$$

of ∞-groupoids. □

We can now proceed to the realization problem. Fix $\Lambda \in \mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathcal{P}}$. Call $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ a *potential n-stage* for Λ when X is n-truncated, $\pi_0 X \simeq \Lambda$, and $X_{S^1} \to X^{S^1}$ is an $(n-1)$-equivalence over X. Let $\mathcal{M}_n(\Lambda)$ be the space of potential n-stages for Λ. Truncation defines $\mathcal{M}_n(\Lambda) \to \mathcal{M}_{n-1}(\Lambda)$, and as in [**Pst17**, Proposition 3.8] the limit $\mathcal{M}_\infty(\Lambda)$ is equivalent to the space of realizations of Λ, i.e. the space of $X \in \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ such that $\pi_0 X \simeq \Lambda$. The following facts summarize some properties of n-stages.

Lemma 2.11. Let X be a potential n-stage for Λ, and choose an isomorphism $\pi_0 X \simeq \Lambda$.

(1) The map $X_{S^k} \to X^{S^k}$ is an $(n-k)$-equivalence over X;
(2) $\Pi_k X \simeq \Lambda\langle k \rangle$ for $k \leq n$;
(3) The only nontrivial homotopy Λ-module of $\tau_! X$ is $\Pi_{n+2}\tau_! X \simeq \Lambda\langle n+1 \rangle$.

Proof. (1) This follows from an inductive argument using the Cartesian squares

$$\begin{tikzcd}
X_{S^{k+1}} \arrow[r] \arrow[d] & X \arrow[d] \\
(X_{S^k})_{S^1} \arrow[r] & X_{S^k} \times_X X_{S^1}
\end{tikzcd}$$

for $k \geq 1$; compare the proof of Proposition 2.5.

(2) This follows from (1).

(3) This follows from Theorem 2.3. □

Lemma 2.12. Fix an n-truncated object $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ such that $\pi_0 X \simeq \Lambda$. Then $X \in \mathcal{M}_n(\Lambda)$ if and only if the map $X_{S_1} \to X^{S^1}$ induces an equivalence $(B_X X_{S^1})_{\leq n} \simeq (B_X X^{S^1})_{\leq n}$. □

Proposition 2.15. Suppose given $Y \in \mathcal{M}_{n-1}(\Lambda)$ together with a Cartesian square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \tau^*\Lambda \arrow[d] \\
Y \arrow[r, "f"] & \tau^* B^{n+1}_\Lambda \Lambda\langle n \rangle
\end{tikzcd}.$$

Then $X \in \mathcal{M}_n(\Lambda)$ if and only if f is adjoint to an equivalence $\tau_! Y \simeq B^{n+2}_\Lambda \Lambda\langle n \rangle$.

Proof. Form Cartesian squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tau^* Z \arrow[r] \arrow[d] & \tau^*\Lambda \arrow[d] \\ Y \arrow[r] & \tau^*\tau_! Y \arrow[r] & \tau^* B^{n+1}_\Lambda \Lambda\langle n\rangle \end{tikzcd} .$$

All maps here are π_0-equivalences, and we wish to show that $X \in \mathcal{M}_n(\Lambda)$ if and only if $Z \simeq \Lambda$. Form the Cartesian cube

$$\begin{tikzcd} B_X(X \times_Y Y_{S^1}) \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dr] \arrow[dd] & \\ & B_Y Y_{S^1} \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\ X \arrow[rr] \arrow[dr] & & \tau^* Z \arrow[dr] & \\ & Y \arrow[rr] & & \tau^*\tau_! Y \end{tikzcd} .$$

As $X_{\leq n-1} \simeq Y_{\leq n-1}$, there are equivalences $(X_{S^1})_{\leq n-1} \simeq (Y_{S^1})_{\leq n-1} \simeq (X \times_Y Y_{S^1})_{\leq n-1}$, and so $(B_X X_{S^1})_{\leq n} \simeq (B_X(X \times_Y Y_{S^1}))_{\leq n} \simeq (X \times_{\tau^* Z} X)_{\leq n}$. By Lemma 2.12, it follows that if $Z \simeq \Lambda$, then $X \in \mathcal{M}_n(\Lambda)$. Conversely, if $X \in \mathcal{M}_n(\Lambda)$ then $(X \times_{\tau^* Z} X)_{\leq n} \simeq (X \times_{\pi_0 X} X)_{\leq n}$, from which it follows that $\tau^* Z \simeq \Lambda$. □

Let $\mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n\rangle, n+1))$ be as defined in the beginning of this subsection, only constructed with respect to $\mathrm{Psh}(\mathrm{h}\mathcal{P})$. This space can be identified using Proposition 2.14.

Lemma 2.13. There is an equivalence

$$\mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n\rangle, n+1)) \cong \mathrm{Map}_{\mathrm{h}\mathcal{P}/\Lambda}(\Lambda; B^{n+2}_\Lambda \Lambda\langle n\rangle)_{\mathrm{Aut}(\Lambda,\Lambda\langle n\rangle)}.$$

Under this equivalence, $B^{n+1}_\Lambda \Lambda\langle n\rangle \in \mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n\rangle, n+1))$ is sent to the zero section. □

This is already enough for a coarse obstruction theory.

Proposition 2.16. Fix $\Lambda \in \mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathcal{P}}$, and let Y be an $(n-1)$-stage for Λ. Then there is an obstruction $\epsilon_n(Y) \in \pi_0 \mathrm{Map}_{h\mathcal{P}/\Lambda}(\Lambda; B^{n+2}_\Lambda \Lambda\langle n\rangle)/\mathrm{Aut}(\Lambda, \Lambda\langle n\rangle)$ which vanishes if and only if there is an n-stage X such that $X_{\leq n-1} \simeq Y$.

Proof. Lemma 2.11 gives us a map $\tau_!\colon \mathcal{M}_{n-1}(\Lambda) \to \mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n\rangle, n+1))$. Let $\epsilon_n(Y)$ be the path component of the image of Y under this map and the equivalence of Lemma 2.13. We conclude by Proposition 2.15. □

The more refined statement is the following.

Theorem 2.12 ([**Pst17**, Theorem 3.15]). For each $n \geq 1$, there is a Cartesian square

$$\begin{tikzcd} \mathcal{M}_n(\Lambda) \arrow[r] \arrow[d] & B\mathrm{Aut}(\Lambda, \Lambda\langle n\rangle) \arrow[d] \\ \mathcal{M}_{n-1}(\Lambda) \arrow[r] & \mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n\rangle, n+1)) \end{tikzcd}.$$

Proof. If $\mathcal{M}_{n-1}(\Lambda)$ is empty, then $\mathcal{M}_n(\lambda)$ is also empty and there is nothing left to show. Otherwise, by choosing $Y \in \mathcal{M}_{n-1}(\Lambda)$ and declaring $F = \{Y\} \times_{\mathcal{M}_{n-1}(\Lambda)} \mathcal{M}_n(\Lambda)$ to be the space of $X \in \mathcal{M}_n(\Lambda)$ equipped with an equivalence $X_{\leq n-1} \simeq Y$, we reduce to verifying that the bottom square in

$$\begin{tikzcd} \mathrm{Eq}(\tau_! Y, B^{n+1}_\Lambda \Lambda\langle n\rangle) \arrow[r] \arrow[d] & \{\Lambda\langle n\rangle\} \arrow[d] \\ F \arrow[r] \arrow[d] & B\mathrm{Aut}(\Lambda, \Lambda\langle n\rangle) \arrow[d] \\ \{\tau_! Y\} \arrow[r] & \mathcal{M}^h(\Lambda +_\Lambda (\Lambda\langle n\rangle, n+1)) \end{tikzcd}$$

is Cartesian. Here the top left space is the space of equivalences $\tau_! Y \simeq B^{n+1}_\Lambda \Lambda\langle n\rangle$. The outer square is Cartesian by definition, so it is sufficient to verify that the top square is Cartesian. Let F' be the space of all $X \in \mathcal{M}(Y +_\Lambda (\Lambda\langle n\rangle, n))$ equipped with an equivalence $X_{\leq n-1} \simeq Y$, so that F is a collection of path components of F', and F' fits into a Cartesian square

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & \mathcal{M}(Y +_\Lambda (\Lambda\langle n\rangle, n)) \arrow[d] \\ \{Y\} \times B\mathrm{Aut}(\Lambda, \Lambda\langle n\rangle) \arrow[r] & \mathcal{M}(Y) \times B\mathrm{Aut}(\Lambda, \Lambda\langle n\rangle) \end{tikzcd}.$$

Form the diagram

$$\begin{tikzcd} \mathrm{Eq}(\tau_! Y, B^{n+1}_\Lambda \Lambda\langle n\rangle) \arrow[r] \arrow[d] & \mathrm{Map}^{0\text{-Eq}}_{\mathcal{P}}(Y, \tau^* B^{n+1}_\Lambda \Lambda\langle n\rangle) \arrow[r] \arrow[d] & \{\Lambda\langle n\rangle\} \arrow[d] \\ F \arrow[r] & F' \arrow[r] & B\mathrm{Aut}(\Lambda, \Lambda\langle n\rangle) \end{tikzcd},$$

where the upper left horizontal map is obtained via the adjunction

$$\mathrm{Map}_{\mathrm{h}\mathcal{P}}(\tau_! Y, B^{n+1}_\Lambda \Lambda\langle n\rangle) \simeq \mathrm{Map}_{\mathcal{P}}(Y, \tau^* B^{n+1}_\Lambda \Lambda\langle n\rangle).$$

The rightmost square is Cartesian by Proposition 2.14, so to show the outer square is Cartesian it is sufficient to verify that the left square is Cartesian. This follows from Proposition 2.15. $\square$

2.5. Localizations and completions

If R is an $\mathbb{E}_2$-ring and $I \subset R_0$ is a finitely generated ideal, then there is a good notion of I-completeness for R-modules, and one can proceed to consider various algebraic structures built from I-complete R-modules. We would like to be able to apply our machinery in this setting; in fact this was our original motivation for working with infinitary theories.

If $\mathrm{LMod}_R^{\mathrm{Cpl}(I)}$ is the category of I-complete left R-modules, and $\mathcal{P} = \mathrm{LMod}_R^{\mathrm{Cpl}(I),\mathrm{free}}$ is the category of I-completions of free R-modules, then to apply our machinery we would like to say $\mathrm{Model}_{\mathcal{P}}^{\Omega} \simeq \mathrm{LMod}_R^{\mathrm{Cpl}(I)}$, and to identify $\mathrm{Model}_{\mathrm{h}\mathcal{P}}$ as something recognizable in terms of LMod_{R_*}. The former always holds, and the latter is possible under a minor algebraic condition on the ideal I. We describe this condition in Subsection 2.5.2 in the more general setting of R-linear theories for a connective $\mathbb{E}_2$-ring R. Before this, in Subsection 2.5.1 we consider some general aspects of the interaction between localizations and theories.

2.5.1. Localizations of theories. We begin with some facts about localizing monads in a 1-categorical setting.

Lemma 2.14. Let $\mathcal{C}$ be a 1-category, $L\colon \mathcal{C} \to \mathcal{C}$ be a localization, and $T\colon \mathcal{C} \to \mathcal{C}$ be a monad. If $LTC \to LTLC$ is an equivalence for all $C \in \mathcal{C}$, then the composite $LTLT \simeq LTT \to LT$ equips LT with the structure of a monad. Moreover, L canonically lifts to a localization $L\colon \mathcal{A}\mathrm{lg}_T \to \mathcal{A}\mathrm{lg}_{LT}$ exhibiting $\mathcal{A}\mathrm{lg}_{LT}$ as the category of T-algebras whose underlying object of $\mathcal{C}$ is L-local.

Proof. This is a diagram chase; see for instance [**Rez18**, Proposition 11.5]. $\square$

Remark 2.8. We fully expect that Lemma 2.14 holds even when $\mathcal{C}$ is an ∞-category. As we do not have a reference, we will avoid arguments that would rely on this. $\triangleleft$

The monad structure on LT obtained by Lemma 2.14 is essentially unique, in the following sense.

Lemma 2.15. Let $\mathcal{C}$ be a 1-category, and let $L\colon \mathcal{C} \to \mathcal{C}$ a localization. Let $T\colon \mathcal{C} \to \mathcal{C}$ be a functor, and suppose that $T, LT \in \mathrm{Fun}(\mathcal{C},\mathcal{C})$ are equipped with right-unital pairings (η,μ) and $(\widehat{\eta},\widehat{\mu})$ in such a way that $T \to LT$ preserves this structure.

(1) For any $C \in \mathcal{C}$, the map $LTC \to LTLC$ is an equivalence;
(2) The pairing on LT is given by the composite $LT \circ LT \xleftarrow{\simeq} LT \circ T = L(T \circ T) \to LT$.

Proof. (1) Write I for the identity on $\mathcal{C}$ and $c\colon I \to L$ for the unit. By naturality of c, the diagram

$$\begin{tikzcd} I \arrow[r, "c"] \arrow[d, "\eta"] & L \arrow[d, "L\eta"] \\ T \arrow[r, "cT"] & LT \end{tikzcd}$$

commutes, and the assumption that c is compatible with units implies that the composite is $\widehat{\eta}$. As a consequence, there is a commutative diagram

$$\begin{tikzcd} LT \arrow[r, "LT\widehat{\eta}"] \arrow[d, "LTc"'] & LTLT \arrow[r, "\widehat{\mu}"] & LT \arrow[d, "LTc"] \\ LTL \arrow[ur, "LTL\eta"'] \arrow[rr, "g"] & & LTL \end{tikzcd},$$

where g is defined so the diagram commutes. To show that LTc is an equivalence, it is sufficient to verify that g is the identity, for then an inverse is given by $\widehat{\mu} \circ LTL\eta$. Indeed, consider the diagram

$$\begin{tikzcd} LTL \arrow[r, "LTLc=LTcL"] \arrow[d, "LTL\eta"] & LTLL \arrow[d, "LTL\eta L"] \\ LTLT \arrow[r, "LTLTc"] \arrow[d, "\widehat{\mu}"] & LTLTL \arrow[d, "\widehat{\mu}L"] \\ LT \arrow[r, "LTc"] & LTL \end{tikzcd}.$$

The top square commutes by naturality of η, and the bottom square commutes by naturality of $\widehat{\mu}$. The clockwise composite is the identity as $L\eta \circ c = \widehat{\eta}$ implies $LTL\eta L \circ LTcL = LT\widehat{\eta}L$, and the counterclockwise composite is g, hence g is the identity.

(2) Observe that the diagram

$$\begin{tikzcd} TT \arrow[r, "cTT"] \arrow[d, "\mu"] \arrow[rr, bend left, "cTcT"] & LTT \arrow[r, "LTcT"] \arrow[d, "L\mu"] & LTLT \arrow[d, "\widehat{\mu}"] \\ T \arrow[r, "cT"] & LT \arrow[r, "="] & LT \end{tikzcd}$$

commutes. Indeed, the leftmost square commutes by naturality of c, and the outermost by compatibility of c with the pairings. As L is a localization, the rightmost square commutes because the outer square commutes. Consider the diagram

$$\begin{tikzcd} LTLT \arrow[r, "LTL\eta T"] \arrow[d, "\widehat{\mu}"] & LTLTT \arrow[d, "\widehat{\mu}T"] \\ LT & LTT \arrow[l, "L\mu"] \arrow[ul, "LTcT"'] \end{tikzcd}.$$

By our proof of (1), the clockwise composite is exactly the pairing $LTLT \simeq LTT \to LT$, so we must verify the outer square commutes. The left triangle commutes by the above, and the right triangle gives the identity on $LTLT$ by our proof of (1), so the outer square indeed commutes. $\square$

Now let $\mathcal{P}$ be a theory, and let $L\colon \mathrm{Model}_{\mathcal{P}} \to \mathrm{Model}_{\mathcal{P}}$ be a localization which preserves geometric realizations, so that the category of L-local objects is realized by restriction $\mathrm{Model}_{L\mathcal{P}} \to \mathrm{Model}_{\mathcal{P}}$; see Proposition 2.7. Let T be a monad on $\mathrm{Model}_{\mathcal{P}}$ which preserves geometric realizations, so that $\mathcal{A}\mathrm{lg}_T \simeq \mathrm{Model}_{T\mathcal{P}}$.

Proposition 2.17. Fix notation as in the preceding paragraph, and suppose that $LTh(P)$ is a T-algebra for each $P \in \mathcal{P}$ naturally in $Th(P)$.

(1) The map $LT \to LTL$ is a natural isomorphism;
(2) The functor LT carries the structure of a monad, informally described by $LTLT \simeq LTT \to LT$;
(3) The localization L lifts to a localization $L\colon \mathcal{A}\mathrm{lg}_T \to \mathcal{A}\mathrm{lg}_{LT}$ realizing $\mathcal{A}\mathrm{lg}_{LT}$ as the category of T-algebras whose underlying object of $\mathrm{Model}_{\mathcal{P}}$ is L-local.

Proof. (1–2) As each $LTh(P)$ is a T-algebra, we can let $LT\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ be the full subcategory of such objects. There is then a commutative diagram

$$\begin{tikzcd}
\mathrm{Model}_{LT\mathcal{P}} \arrow[r] \arrow[d] & \mathrm{Model}_{T\mathcal{P}} \arrow[d] \\
\mathrm{Model}_{L\mathcal{P}} \arrow[r] & \mathrm{Model}_{\mathcal{P}}
\end{tikzcd}$$

of restriction functors which preserve geometric realizations and which are the forgetful functors of monadic adjunctions. The monad associated to $\mathrm{Model}_{LT\mathcal{P}} \to \mathrm{Model}_{\mathcal{P}}$ has underlying functor LT, and this gives a map $T \to LT$ of monads. We conclude by applying Lemma 2.15 to the homotopy category of $\mathrm{Model}_{\mathcal{P}}$.

(3) Here we are claiming that the above diagram is Cartesian, and that

$$\begin{tikzcd}
\mathrm{Model}_{LT\mathcal{P}} \arrow[d] & \mathrm{Model}_{T\mathcal{P}} \arrow[l] \arrow[d] \\
\mathrm{Model}_{L\mathcal{P}} & \mathrm{Model}_{\mathcal{P}} \arrow[l, "L"']
\end{tikzcd}$$

commutes. The latter is clear, as can be checked on objects of the form $T(P)$ for $P \in \mathcal{P}$, and this implies the former. $\square$

It is only a bit of extra work to include loop theories in the story.

Proposition 2.18. Let $\mathcal{P}$ be a loop theory, and let $L\colon \mathrm{Model}_{\mathcal{P}} \to \mathrm{Model}_{\mathcal{P}}$ be a localization which preserves geometric realizations. Suppose that $Lh(P) \in \mathrm{Model}_{\mathcal{P}}^{\Omega}$ for each $P \in \mathcal{P}$. Then

(1) $\mathcal{M}\mathrm{odel}^{\Omega}_{L\mathcal{P}}$ is a localization of $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$;

(2) The diagram

$$\begin{tikzcd} \mathcal{M}\mathrm{odel}^{\Omega}_{L\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}} \arrow[d] \\ \mathcal{M}\mathrm{odel}_{L\mathcal{P}} \arrow[r] & \mathcal{M}\mathrm{odel}_{\mathcal{P}} \end{tikzcd}$$

is Cartesian.

Proof. (1) Because $Lh(P) \in \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ for each $P \in \mathcal{P}$, the full subcategory $L\mathcal{P} \subset \mathcal{M}\mathrm{odel}_{L\mathcal{P}}$ is a loop theory, and restriction gives $\mathcal{M}\mathrm{odel}^{\Omega}_{L\mathcal{P}} \to \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$. This is fully faithful, being obtained from $\mathcal{M}\mathrm{odel}_{L\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, and is the forgetful functor of a monadic adjunction by Proposition 2.6.

(2) Here we are claiming that $\mathcal{M}\mathrm{odel}^{\Omega}_{L\mathcal{P}}$ consists of those objects of $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ which are L-local in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$, which is now clear. □

Proposition 2.19. Let $\mathcal{P}$ be a loop theory, and let $L\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ be a localization which preserves geometric realizations such that $Lh(P) \in \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ for each $P \in \mathcal{P}$. Let T be a monad on $\mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$, and suppose that

(a) T satisfies the criteria of Theorem 2.5, so that $\mathcal{A}\mathrm{lg}_T \simeq \mathcal{M}\mathrm{odel}^{\Omega}_{T\mathcal{P}}$;

(b) $LTh(P) \in \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ for each $P \in \mathcal{P}$;

(c) $LTh(P)$ is a T-algebra for each $P \in \mathcal{P}$ naturally in $Th(P)$.

Then

(1) The functor LT carries the structure of a monad, and the associated forgetful functor can be identified as restriction $\mathcal{M}\mathrm{odel}^{\Omega}_{LT\mathcal{P}} \to \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$;

(2) The square

$$\begin{tikzcd} \mathcal{M}\mathrm{odel}^{\Omega}_{LT\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{M}\mathrm{odel}^{\Omega}_{T\mathcal{P}} \arrow[d] \\ \mathcal{M}\mathrm{odel}^{\Omega}_{L\mathcal{P}} \arrow[r] & \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}} \end{tikzcd}$$

is Cartesian;

(3) The square

$$\begin{tikzcd} \mathcal{M}\mathrm{odel}^{\Omega}_{LT\mathcal{P}} \arrow[d] & \mathcal{M}\mathrm{odel}^{\Omega}_{T\mathcal{P}} \arrow[l] \arrow[d] \\ \mathcal{M}\mathrm{odel}^{\Omega}_{L\mathcal{P}} & \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}} \arrow[l, "L"'] \end{tikzcd}$$

commutes.

Proof. The restriction $\mathcal{M}\mathrm{odel}^{\Omega}_{LT\mathcal{P}} \to \mathcal{M}\mathrm{odel}^{\Omega}_{T\mathcal{P}}$ is the forgetful functor of a monadic adjunction by Proposition 2.6. It is obtained from $\mathcal{M}\mathrm{odel}_{LT\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{T\mathcal{P}}$, so is the inclusion of a reflective subcategory by Proposition 2.17. We claim that the associated localization is a lift of L; (2) follows quickly. This is the content of (3), and moreover shows that the monad associated to $\mathcal{M}\mathrm{odel}^{\Omega}_{LT\mathcal{P}} \to \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}}$ has underlying functor LT, proving (1). Consider the cube

$$\begin{tikzcd}
\mathcal{M}\mathrm{odel}^{\Omega}_{LT\mathcal{P}} \arrow[dd] & & \mathcal{M}\mathrm{odel}^{\Omega}_{T\mathcal{P}} \arrow[ll, dashed] \arrow[dr] \arrow[dd] & \\
& \mathcal{M}\mathrm{odel}_{LT\mathcal{P}} \arrow[ul] \arrow[dd] & & \mathcal{M}\mathrm{odel}_{T\mathcal{P}} \arrow[ll] \arrow[dd] \\
\mathcal{M}\mathrm{odel}^{\Omega}_{L\mathcal{P}} & & \mathcal{M}\mathrm{odel}^{\Omega}_{\mathcal{P}} \arrow[ll, dashed] \arrow[dr] & \\
& \mathcal{M}\mathrm{odel}_{L\mathcal{P}} \arrow[ul] & & \mathcal{M}\mathrm{odel}_{T\mathcal{P}} \arrow[ll]
\end{tikzcd},$$

consisting of restrictions or left adjoints. The dashed arrows are such that the top and bottom faces commute, and the back face is the square of (3). As the rest of the diagram commutes, so does the back face, as claimed. $\square$

2.5.2. ***R*-linear theories and completions.** We now consider the main cases of interest, namely those derived from I-completion. We begin by reviewing the relevant notion of completeness; this story is developed in [**Lur18**, Section 7], and our perspective is strongly influenced by [**Rez18**].

Fix a connective $\mathbb{E}_2$-ring R. In [**Lur18**, Definition D.1.4.1], the notion of an R-linear prestable category is introduced. In short, where $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{sfg}}$ is the category of left R-modules equivalent to $R^{\oplus n}$ for some $n < \infty$, an R-linear structure on a presentable stable category $\mathcal{M}$ is equivalent to an additive and monoidal functor $\mathrm{L}\mathcal{M}\mathrm{od}_R^{\mathrm{sfg}} \to \mathrm{Fun}^L(\mathcal{M}, \mathcal{M})$, where the latter is the category of colimit-preserving endofunctors of $\mathcal{M}$. For convenience, we extend this definition to allow $\mathcal{M}$ to be an arbitrary presentable additive category; we will only apply the theory of [**Lur18**] beyond the definitions in the case where $\mathcal{M}$ is stable.

Let $I \subset R_0$ be a finitely generated ideal, and let $\mathcal{M}$ be an R-linear stable category. An object $M \in \mathcal{M}$ is said to be *I-nilpotent* if $R[x^{-1}] \otimes_R M \simeq 0$ for all $x \in I$, is said to be *I-local* if $\mathrm{Map}_{\mathcal{M}}(N, M)$ is contractible for all I-nilpotent N, and is said to be *I-complete* if $\mathrm{Map}_{\mathcal{M}}(N, M)$ is contractible for all I-local N. Let $\mathcal{M}^{\mathrm{Cpl}(I)} \subset \mathcal{M}$ denote the full subcategory of I-complete objects. Then $\mathcal{M}^{\mathrm{Cpl}(I)}$ is a reflective subcategory of $\mathcal{M}$, with associated localization the functor $M \mapsto M_I^{\wedge}$ of *I-completion*.

We will need an explicit formula for I-completion, and for this we must first fix some notation. Let $\underline{h} = \{1, \ldots, h\}$, and let $P(\underline{h})$ denote the powerset of $\underline{h}$, so that an h-cube is given by a functor from $P(\underline{h})$. Given an h-cube $V \colon P(\underline{h}) \to \mathcal{C}$ in some category $\mathcal{C}$, we will for $i \notin S \subset \underline{h}$ write $V_i \colon V(S) \to V(S \cup \{i\})$ for the map induced by $S \subset S \cup \{i\}$. Given an h-cube V in a category $\mathcal{C}$ with finite colimits, write $\operatorname{tCof} V$ for the total cofiber of V.

Suppose now that I is a finitely generated ideal, and make a choice of generators $\underline{u} = (u_1, \ldots, u_h)$. If M is any object of a category with countable products on which $u_1, \ldots, u_h$ act, we can define the h-cube

$$K(M; \underline{u}) \colon P(\underline{h}) \to \mathcal{M}, \qquad S \mapsto M[[T_1, \ldots, T_h]] = M^{\times \omega^h},$$

where

$$K(M; \underline{u})_i = (T_i - u_1) \colon M[[T_1, \ldots, T_h]] \to M[[T_1, \ldots, T_h]].$$

Proposition 2.20. Let $\mathcal{M}$ be an R-linear stable category. Then for $M \in \mathcal{M}$, there is an equivalence

$$M_I^\wedge \simeq \operatorname{tCof} K(M; \underline{u}).$$

Proof. When $h = 1$, this is a reformulation of [**Lur18**, Proposition 7.3.2.1]. The general case then follows from [**Lur18**, Proposition 7.3.3.2]. ☐

Let $\mathcal{P}$ be an additive theory. Say that $\mathcal{P}$ is an *R-linear theory* if we have chosen an additive monoidal functor $\mathcal{M}\mathrm{od}_R^{\mathrm{sfg}} \to \operatorname{Fun}^{\oplus}(\mathcal{P}, \mathcal{P})$, where the latter is the category of coproduct-preserving endofunctors of $\mathcal{P}$. If $\mathcal{P}$ is an additive loop theory, say that $\mathcal{P}$ is an R-linear loop theory if we have chosen an additive monoidal functor $\mathcal{M}\mathrm{od}_R^{\mathrm{sfg}} \to \operatorname{Fun}^{\oplus,\Sigma}(\mathcal{P}, \mathcal{P})$, where the latter is the category of coproduct and suspension-preserving endofunctors of $\mathcal{P}$. Note that if $\mathcal{P}$ is an R-linear theory, then so is $\mathrm{h}\mathcal{P}$. If $\mathcal{P}$ is an R-linear theory, then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is an R-linear category, and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ is an R-linear stable category; if $\mathcal{P}$ is an R-linear loop theory, then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is an R-linear category, stable so long as $\mathcal{P}$ is a stable loop theory.

Proposition 2.21. Let $\mathcal{P}$ be an R-linear theory. Then I-completion

$$\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}, \qquad X \mapsto X_I^\wedge$$

restricts to a localization of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ which preserves geometric realizations. This localization is given explicitly by

$$X_I^\wedge = \operatorname{tCof} K(X; \underline{u})$$

for $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

Proof. This follows from the description of I-completion given in Proposition 2.20. ☐

Call $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ *I-complete* if $X \simeq X_I^\wedge$. If $\mathcal{P}$ is a stable loop theory, then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is itself a stable R-linear category, so there is a possible ambiguity in speaking of I-complete objects of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$. However, this ambiguity turns out to vanish.

Lemma 2.16. Let $\mathcal{P}$ be an R-linear stable loop theory, and fix $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$. Then X is I-complete as an object of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ if and only if it is I-complete as an object of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

Proof. If $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ is I-complete in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$, then $X \simeq \mathrm{tCof}(X;\underline{u})$ in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$. So this total cofiber lives in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, and thus X is I-complete in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$.

Suppose conversely that X is I-complete in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$. By definition, X is I-complete in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ if and only if for every $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ which is I-local, the mapping space $\mathrm{Map}_{\mathcal{P}}(M,X)$ is contractible. As $\mathrm{Map}_{\mathcal{P}}(M,X) \simeq \mathrm{Map}_{\mathcal{P}}(LM,LX)$ where $L\colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$ is the localization, and as LX is I-complete in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\Omega}$, it is sufficient to verify that L preserves I-local objects. This is a consequence of [**Lur18**, Proposition 7.2.4.9]. $\square$

Proposition 2.22. Let $\mathcal{P}$ be an R-linear stable looptheory. Let $\mathcal{P}_I^{\wedge} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$ be the full subcategory spanned by the I-completions of objects of $\mathcal{P}$. Then

$$\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega,\mathrm{Cpl}(I)} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}_I^{\wedge}}^{\Omega}.$$

Proof. By Theorem 2.5, it is sufficient to verify that

$$\begin{tikzcd} \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \arrow[r, "(-)_I^{\wedge}"] \arrow[d] & \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega} \\ \mathcal{M}\mathrm{odel}_{\mathcal{P}} \arrow[r, "(-)_I^{\wedge}"] & \mathcal{M}\mathrm{odel}_{\mathcal{P}} \arrow[u, "L"'] \end{tikzcd}$$

commutes, which follows from Proposition 2.20. $\square$

Some additional hypotheses are necessary to make practical use of Proposition 2.22. For example, $\mathrm{h}\mathcal{P}$ is itself an R-linear theory, so one would like to identify $\mathcal{M}\mathrm{odel}_{\mathrm{h}(\mathcal{P}_I^{\wedge})} \simeq \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\mathrm{Cpl}(I)}$; however, this is not true in general. Determining when properties such as this hold amount to understanding when, given $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, the completion $X_I^{\wedge}$ as computed in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ still lives in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$. This turns out to be an essentially algebraic condition.

Call a theory $\mathcal{P}$ *pretame* if $\tau_!\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$ preserves countable products. It follows from Theorem 2.3 that every loop theory is pretame. The purpose of the pretameness condition is the following.

Lemma 2.17. Let $\mathcal{P}$ be a pretame R-linear theory. Then

$$\tau_!(X_I^{\wedge}) = (\tau_! X)_I^{\wedge}$$

for all $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

Proof. As $\mathcal{P}$ is pretame, there is an equivalence $\tau_! K(X;\underline{u}) \simeq K(\tau_! X;\underline{u})$ of h-cubes, so the claim follows from Proposition 2.21. $\square$

If $\mathcal{P}$ is a discrete R-linear theory, then the I-completion of discrete objects of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ admits an algebraic description. Given an abelian category $\mathcal{A}$, an h-cube $V\colon P(\underline{h}) \to \mathcal{A}$ may be viewed as an h-dimensional complex, and so we may form the total complex C_*V. This satisfies $H_0(C_*V) = \mathrm{tCof}\, V$, this total cofiber taken in the 1-category $\mathcal{A}$. In general, C_*V is a model of the derived total cofiber of V in the following sense.

Lemma 2.18. Let $\mathcal{A}$ be an abelian category with enough projectives, and let $V\colon P(\underline{h}) \to \mathcal{A}$ be an h-cube. Then $(\mathbb{L}_n\mathrm{tCof})(V) = H_n(C_*V)$. ☐

In particular, if $\mathcal{A}$ is an R_0-linear abelian category with countable products and $M \in \mathcal{A}$, then we can form the chain complex $C_*K(M;\underline{u})$. In this case, set $K_n(M;\underline{u}) = H_nC_*K(M;\underline{u})$.

Lemma 2.19. Let $\mathcal{P}$ be a discrete R-linear theory. For $M \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ discrete, there are isomorphisms

$$\pi_*(M_I^\wedge) \cong K_*(M;\underline{u}).$$

In particular,

(1) $M_I^\wedge$ is h-truncated, h being the length of the sequence $\underline{u}$;
(2) $M_I^\wedge$ is discrete if and only if $K_n(M;\underline{u}) = 0$ for $n > 0$.

Proof. This is an immediate consequence of Proposition 2.21 and Lemma 2.18. ☐

We now arrive at the promised characterization.

Proposition 2.23. Let $\mathcal{P}$ be a pretame R-linear theory, and fix $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$. Suppose that $\tau_! X = \pi_0 X$. Then the following are equivalent:

(1) The object $\tau_! X_I^\wedge$ of $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$ is discrete;
(2) The object $K_n(\pi_0 X;\underline{u})$ of $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\heartsuit}$ vanishes for $n > 0$.

If $\mathcal{P}$ is a loop theory, then $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$, and these are equivalent to:

(3) The completion $X_I^\wedge$ as computed in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ lives in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega}$.

Proof. The equivalence of (1) and (2) follows from Lemma 2.17 and Lemma 2.19, and the inclusion of (3) follows from Corollary 2.1. ☐

Say that I is *tame* on $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ if the equivalent conditions of Proposition 2.23 hold for X, and say that I is *tame* on $\mathcal{P}$ if I is tame on $h(P)$ for $P \in \mathcal{P}$. In particular, I is tame on $\mathcal{P}$ if and only if it is tame on $\mathrm{h}\mathcal{P}$, i.e. tameness is an algebraic condition.

If S is an R-algebra and $\mathcal{P} = \mathrm{L}\mathcal{M}\mathrm{od}_S^{\mathrm{free}}$, then I is tame on $\mathcal{P}$ precisely when it is tame on $\mathrm{L}\mathcal{M}\mathrm{od}_{S_*}^{\mathrm{free}} = \mathrm{h}(\mathrm{L}\mathcal{M}\mathrm{od}_S^{\mathrm{free}})$. Tameness in this setting coincides with the notion of tameness discussed in Greenlees-May [**GM92**] [**GM95**] and Rezk [**Rez18**, Section 8], and holds in a number of situations. For example, if M is an S_0-module, then I is tame on M when I is generated by a sequence which is regular on M, or when $M = N^{\oplus J}$ for some Noetherian S_0-module N and some set J [**Lur18**, Corollary 7.3.6.1].

The definition of tameness is chosen so that the following holds.

THEOREM 2.13. Let $\mathcal{P}$ be a stable R-linear theory, and suppose that I is tame on $\mathcal{P}$. Let $\mathcal{P}_I^\wedge \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^\Omega$ be the full subcategory spanned by the I-completions of objects of $\mathcal{P}$. Then there are equivalences

$$\mathcal{M}\mathrm{odel}_{\mathcal{P}_I^\wedge}^\Omega \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\Omega,\mathrm{Cpl}(I)}, \qquad \mathcal{M}\mathrm{odel}_{\mathcal{P}_I^\wedge} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\mathrm{Cpl}(I)}, \qquad \mathcal{M}\mathrm{odel}_{\mathrm{h}(\mathcal{P}_I^\wedge)} \simeq \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}^{\mathrm{Cpl}(I)}.$$

PROOF. The first equivalence is a restatement of Proposition 2.22. For the remaining two, observe that as I is tame on $\mathcal{P}$, the category $\mathcal{P}_I^\wedge$ may be identified as the full subcategory of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ spanned by $h(P)_I^\wedge$ for $P \in \mathcal{P}$, where this completion is taken in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$, and that $\mathrm{h}(\mathcal{P}_I^\wedge) \subset \mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{P}}$ may be identified as the full subcategory spanned by $(\pi_0 h(P))_I^\wedge$ for $P \in \mathcal{P}$. So these equivalences are consequences of Proposition 2.21 and Proposition 2.7. $\square$

Theorem 2.13 extends by combination with Subsection 2.5.1 to describe unstable theories built out of stable theories in completed settings. We will see a more explicit application of these ideas in and around Subsection 4.2.4.

2.6. Spectral sequences

This section gives the facts that were needed in Section 2.3 about towers in a stable category with t-structure and their associated spectral sequences. We will freely use material and notation from [**Lur17a**, Section 1.2].

2.6.1. Construction and convergence. Fix a stable category $\mathcal{C}$ with t-structure, and let $\mathcal{A}$ be the heart of $\mathcal{C}$. There results a functor $\pi_0 = \tau_{\leq 0}\tau_{\geq 0} \colon \mathcal{C} \to \mathcal{A}$, and we set $\pi_p = \pi_0 \circ \Sigma^{-p}$. Fix a tower

$$X = \cdots \to X(-1) \to X(0) \to X(1) \to \cdots$$

in $\mathcal{C}$. Following [**Lur17a**, Section 1.2], there is for each $-\infty \leq p \leq q$ an object $X(p,q)$ in $\mathcal{C}$, where $X(-\infty,p) = X(p)$ and for $p \leq q \leq r$ there is a chosen cofiber sequence

$$X(p,q) \xrightarrow{\eta} X(p,r) \xrightarrow{\eta} X(q,r) \ .$$

In particular, $X(p,q)$ sits in a cofiber sequence

$$X(p) \xrightarrow{\eta} X(q) \xrightarrow{\eta} X(p,q) \ .$$

Define

$$E^r_{p,q} = \mathrm{Im}\left(\pi_q X(p-r,p) \to \pi_q X(p-1,p+r-1)\right);$$

we abbreviate $E^r_{p,*}$ as E^r_p when it simplifies the notation. Using the diagrams

$$
\begin{array}{ccc}
X(p-r,p) & \xrightarrow{\eta} & X(p-1,p+r-1) \\
\downarrow & & \downarrow \\
\Sigma X(p-2r,p-r) & \xrightarrow{\Sigma\eta} & \Sigma X(p-r-1,p-1)
\end{array},
$$

we obtain maps

$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q-1}.$$

PROPOSITION 2.24 ([**Lur17a**, Proposition 1.2.2.7]). With notation as above,

(1) $d^r \circ d^r = 0$;

(2) There are canonical equivalences $E^{r+1} = H(E^r, d^r)$.

In particular, $\{E^r, d^r\}$ is a spectral sequence of objects of $\mathcal{A}$. □

Write $E(X)$ for this spectral sequence. We would like to identify some simple criteria for convergence. Suppose that $\mathcal{C}$ and $\mathcal{A}$ admit countable direct sums, and thus all countable colimits. For a tower X, write $X(\infty) = \operatorname{colim}_{p\to\infty} X(p)$. The following may be proved just as in [**Lur17a**, Proposition 1.2.2.14].

PROPOSITION 2.25. Fix a tower X, and suppose

(a) The connectivity of $X(p)$ goes to ∞ as p goes to $-\infty$;

(b) $\operatorname{colim}_{r\to\infty} \pi_* X(p,p+r) \cong \pi_* \operatorname{colim}_{r\to\infty} X(p,p+r)$ for all $p \in \mathbb{Z} \cup \{-\infty\}$;

and moreover one of the following holds:

(c) The t-structure on $\mathcal{C}$ is compatible with filtered colimits;

(c′) For all $q \in \mathbb{Z}$, the map $\pi_q X(p) \to \pi_q X(p+1)$ is an isomorphism for all but finitely many p.

Then $E(X)$ converges to $\pi_* X(\infty)$. Explicitly, if $A_q = \pi_q X(\infty)$, then

(1) For all fixed p, q and all sufficiently large r, there are canonical inclusions $E^r_{p,q} \subset E^{r+1}_{p,q}$, and in case (c′) these eventually stabilize;

(2) Where $F^p A_q = \operatorname{Im}(\pi_q X(p) \to \pi_q X(\infty))$, both $F^p A_q = 0$ for p sufficiently small and $\cup_p F^p A_q = A_q$, and in case (c′) this filtration is finite;

(3) There are canonical isomorphisms $F^p A_q / F^{p-1} A_q \cong E^\infty_{p,q}$. □

2.6.2. Monoidal properties of towers. Fix a stable category $\mathcal{C}$ with t-structure, and let $\mathcal{O}$ be a single-colored ∞-operad. Following [**Lur17a**, Definition 2.2.1.6], say an $\mathcal{O}$-monoidal structure on $\mathcal{C}$ is *compatible with the t-structure* on $\mathcal{C}$ if it respects finite colimits, and for all $f \in \mathcal{O}(n)$, the tensor product $\otimes_f$ sends $\mathcal{C}_{\geq 0}^{\times n}$ into $\mathcal{C}_{\geq 0}$. Fix such an $\mathcal{O}$-monoidal structure on $\mathcal{C}$.

PROPOSITION 2.26. The functor $\mathcal{C} \to \operatorname{Fun}(\mathbb{Z}, \mathcal{C})$ sending an object to its Whitehead tower is canonically lax $\mathcal{O}$-monoidal.

Proof. This functor factors as the composite of the diagonal $\mathcal{C} \to \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ and the endofunctor W of $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ sending a tower $n \mapsto X(n)$ to the new tower $n \mapsto X(n)_{\geq -n}$. The former is lax $\mathcal{O}$-monoidal, as $\mathbb{Z} \to \{0\}$ is monoidal, hence it is sufficient to verify that the latter is lax $\mathcal{O}$-monoidal. This follows from [**Lur17a**, Proposition 2.2.1.1], for W is a colocalization of $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$, with image closed under the $\mathcal{O}$-monoidal structure by our hypotheses. $\square$

Restrict now to the case where $\mathcal{O}$ is the nonunital $\mathbb{A}_2$-operad. In other words, fix a pairing $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is exact in each variable and sends $\mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0}$ to $\mathcal{C}_{\geq 0}$. Writing $\mathcal{A}$ for the heart of $\mathcal{C}$, this gives a pairing $\overline{\otimes}$ on $\mathcal{A}$ by

$$M \mathbin{\overline{\otimes}} N = \pi_0(M \otimes N).$$

For $X', X'' \in \mathcal{C}$, there is a canonical Künneth map $\pi_0 X' \mathbin{\overline{\otimes}} \pi_0 X'' \to \pi_0(X \otimes X'')$ given by

$$\pi_0 X' \mathbin{\overline{\otimes}} \pi_0 X'' = \pi_0(X'_{\geq 0} \otimes X''_{\geq 0}) \to \pi_0(X' \otimes X'').$$

There is not such a canonical map in nonzero degrees, the issue being the following. As $\otimes$ is exact in each variable, there are canonical isomorphisms $(\Sigma X') \otimes X'' \simeq \Sigma(X' \otimes X'')$ and $X' \otimes (\Sigma X'') \simeq \Sigma(X' \otimes X'')$. However, the diagram

$$\begin{tikzcd}
(\Sigma^{q'} X') \otimes (\Sigma^{q''} X'') \arrow[r] \arrow[d] & \Sigma^{q'}(X' \otimes (\Sigma^{q''} X'')) \arrow[d] \\
\Sigma^{q''}((\Sigma^{q'} X') \otimes X'') \arrow[r] & \Sigma^{q'+q''}(X' \otimes X'')
\end{tikzcd}$$

can only be made to commute up to a switch map $S^{q'+q''} \simeq S^{q'} \otimes S^{q''} \simeq S^{q''} \otimes S^{q'} \simeq S^{q''+q'} = S^{q'+q''}$, and so on π_0 up to a sign of $(-1)^{q'q''}$. For the rest of this section, we choose the isomorphism given by the counterclockwise composite; in other words, we choose

$$(\Sigma^{q'} X') \otimes (\Sigma^{q''} X'') = \Sigma^{q''} \Sigma^{q'} (X' \otimes X'').$$

This choice falls naturally out of the convention of pretending that $(\Sigma X') \otimes X''$ and $\Sigma(X' \otimes X'')$ are the "same", whereas $X' \otimes (\Sigma X'')$ and $\Sigma(X' \otimes X'')$ are "different". Having made a choice, we obtain a natural transformation

$$\begin{aligned}
\pi_{q'} X' \mathbin{\overline{\otimes}} \pi_{q''} X'' &= \pi_0 \Sigma^{-q'} X' \mathbin{\overline{\otimes}} \pi_0 \Sigma^{-q''} X'' \\
&\to \pi_0(\Sigma^{-q'} X' \otimes \Sigma^{-q''} X'') \simeq \pi_{q'+q''}(X' \otimes X'').
\end{aligned}$$

With this choice, the diagram

$$\begin{tikzcd}
\pi_{q'} \Sigma X' \mathbin{\overline{\otimes}} \pi_{q''} X'' \arrow[r, "="] \arrow[d] & \pi_{q'-1} X' \otimes \pi_{q''} X'' \arrow[d] \\
\pi_{q'+q''} \Sigma X' \otimes X'' \arrow[r, "\simeq"] & \pi_{q'+q''-1} X' \otimes X''
\end{tikzcd}$$

commutes, whereas the diagram

$$\begin{tikzcd}
\pi_{q'}X' \mathbin{\overline{\otimes}} \pi_{q''}\Sigma X'' \arrow[r, "="] \arrow[d] & \pi_{q'}X' \mathbin{\overline{\otimes}} \pi_{q''-1}X'' \arrow[d] \\
\pi_{q'+q''}X' \otimes \Sigma X'' \arrow[r, "\simeq"] & \pi_{q'+q''-1}X' \otimes X''
\end{tikzcd}$$

commutes up to a factor of exactly $(-1)^{q'}$. This is the origin of the signs that will appear for us.

We end this subsection by recording a concrete description of a pairing in $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$.

LEMMA 2.20. A pairing $X' \otimes X'' \to X$ in $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ is equivalent to the choice of pairings $X'(p') \otimes X''(p'') \to X(p' + p'')$ for $p', p'' \in \mathbb{Z}$, together with homotopies filling in the cubes

$$\begin{tikzcd}
X'(p'-1) \otimes X''(p''-1) \arrow[rr] \arrow[dr] \arrow[dd] & & X'(p') \otimes X''(p''-1) \arrow[dr] \arrow[dd] & \\
& X(p'+p''-2) \arrow[rr] \arrow[dd] & & X(p'+p''-1) \arrow[dd] \\
X'(p'-1) \otimes X''(p'') \arrow[rr] \arrow[dr] & & X'(p') \otimes X''(p'') \arrow[dr] & \\
& X(p'+p''-1) \arrow[rr] & & X(p'+p'')
\end{tikzcd}.$$

PROOF. This is immediate from the construction of the tensor product in $\mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ and the form of mapping spaces in $\mathrm{Fun}(\mathbb{Z} \times \mathbb{Z}, \mathcal{C})$. $\square$

2.6.3. Pairings of spectral sequences. Fix conventions as in the previous subsections. Our goal in this subsection is to verify that every pairing $X' \otimes X'' \to X$ of towers gives rise to a pairing $E(X') \mathbin{\overline{\otimes}} E(X'') \to E(X)$ of spectral sequences. Before giving the main construction, we point out the following. By a cofibering $X''(-1) \to X''(0) \to C''(0)$, we refer really to the left square in a suitable coherently commutative diagram

$$\begin{tikzcd}
X''(-1) \arrow[r] \arrow[d] & X''(0) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & C''(0) \arrow[r, "\delta"] & \Sigma X''(-1)
\end{tikzcd},$$

from which we obtain the right square, in particular the boundary map δ. As $\otimes$ is exact in both variables, for any $X' \in \mathcal{C}$ original cofiber sequence tensors to a cofiber sequence $X' \otimes X''(-1) \to X' \otimes X''(0) \to X' \otimes C''(0)$. Again, this refers really to the left square in

$$\begin{tikzcd}
X' \otimes X''(-1) \arrow[r] \arrow[d] & X' \otimes X''(0) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X' \otimes C''(0) \arrow[r, "\delta'"] & \Sigma(X' \otimes X''(-1))
\end{tikzcd},$$

where we have implicitly identified $X' \otimes 0 \simeq 0$, and from this we obtain the right square, in particular the boundary map δ'. This diagram is canonically equivalent to the diagram obtained by tensoring the first with X', and so δ' is equivalent to the composite

$$X' \otimes C''(0) \xrightarrow{X' \otimes \delta} X' \otimes \Sigma X''(-1) \xrightarrow{\simeq} \Sigma(X' \otimes X''(-1)) \ .$$

We now proceed to the main construction. Fix the data of cofiberings

$$X(-2) \to X(-1) \to C(-1) \qquad X(-1) \to X(0) \to C(0)$$
$$X'(-1) \to X'(0) \to C'(0) \qquad X''(-1) \to X''(0) \to C''(0),$$

as well as the data of a filled in cube

$$\begin{tikzcd}
X'(-1) \otimes X''(-1) \arrow[rr] \arrow[dd] \arrow[dr] & & X'(0) \otimes X''(-1) \arrow[dd] \arrow[dr] & \\
& X(-2) \arrow[rr] \arrow[dd] & & X(-1) \arrow[dd] \\
X'(-1) \otimes X''(0) \arrow[rr] \arrow[dr] & & X'(0) \otimes X''(0) \arrow[dr] & \\
& X(-1) \arrow[rr] & & X(0)
\end{tikzcd} .$$

Our initial cofiberings, together with the fact that $\otimes$ is exact in each variable, give rise to a canonical isomorphism from the total cofiber of the back face of this cube to $C'(0) \otimes C''(0)$. As a consequence of this, we can form the commutative diagrams

$$\begin{tikzcd}
X'(-1) \otimes X''(0) \cup_{X'(-1) \otimes X''(-1)} X'(0) \otimes X''(-1) \arrow[r] \arrow[d] & X(-1) \arrow[d] \\
X'(0) \otimes X''(0) \arrow[r] \arrow[d] & X(0) \arrow[d] \\
C'(0) \otimes C''(0) \arrow[r, dashed] & C(0)
\end{tikzcd}$$

$$\begin{tikzcd}
X'(-1) \otimes X''(-1) \arrow[r] \arrow[d] & X(-2) \arrow[d] \\
X'(-1) \otimes X''(0) \cup_{X'(-1) \otimes X''(-1)} X'(0) \otimes X''(-1) \arrow[r] \arrow[d, "f"] & X(-1) \arrow[d] \\
X'(-1) \otimes C''(0) \oplus C'(0) \otimes X''(-1) \arrow[r, dashed] & C(-1)
\end{tikzcd} ,$$

where the columns have the structure of cofiber sequences and the bottom squares are induced from this. From the construction of the maps involved, we obtain the following.

Lemma 2.21. In the diagram

$$\begin{tikzcd}
C'(0) \otimes C''(0) \arrow[r] \arrow[d] & C(0) \arrow[d] \\
\Sigma(X'(-1) \otimes X''(0) \cup_{X'(-1)\otimes X''(-1)} X'(0) \otimes X''(-1)) \arrow[r] \arrow[d, "\Sigma f"] & \Sigma X(-1) \arrow[d] \\
\Sigma(X'(-1) \otimes C''(0) \oplus C'(0) \otimes X''(-1)) \arrow[r] & \Sigma C(-1)
\end{tikzcd}$$

obtained from the above data, the left vertical composite is given by the sum of the maps

$$C'(0) \otimes C''(0) \to (\Sigma X'(-1)) \otimes C''(0) \simeq \Sigma(X'(-1) \otimes C''(0))$$
$$C'(0) \otimes C''(0) \to C'(0) \otimes (\Sigma X''(-1)) \simeq \Sigma(C'(0) \otimes X''(-1)).$$

□

We are now in a position to prove the following.

Theorem 2.14. A pairing $X' \otimes X'' \to X$ of towers gives rise to a pairing $E(X') \overline{\otimes} E(X'') \to E(X'')$ of spectral sequences, i.e. pairings

$$\smile\colon E^r_{p',q'}(X') \overline{\otimes} E^r_{p'',q''}(X'') \to E^r_{p'+p'',q'+q''}(X)$$

satisfying the Leibniz rule

$$d^r(x' \smile x'') = d^r(x') \smile x'' + (-1)^{q'} x' \smile d^r(x''),$$

where moreover the pairing on E^r is induced from naturally defined maps

$$X'(p'-r, p') \otimes X''(p''-r, p'') \to X(p'+p''-r, p'+p''),$$

and the pairing on E^{r+1} is induced by that on E^r.

Proof. From the pairing $X' \otimes X'' \to X$, we obtain solid cubes

$$\begin{tikzcd}
X'(p'-r) \otimes X''(p''-r) \arrow[rr] \arrow[dr] \arrow[dd] & & X'(p') \otimes X''(p''-r) \arrow[dr] \arrow[dd] & \\
& X(p'+p''-2r) \arrow[rr] \arrow[dd] & & X(p'+p''-r) \arrow[dd] \\
X'(p'-r) \otimes X''(p'') \arrow[rr] \arrow[dr] & & X'(p') \otimes X''(p'') \arrow[dr] & \\
& X(p'+p''-r) \arrow[rr] & & X(p'+p'')
\end{tikzcd},$$

giving rise to pairings

$$X'(p'-r,p') \otimes X''(p''-r,p'') \to X(p'+p''-r,p'+p'')$$

fitting into commutative diagrams

$$\begin{tikzcd}
X'(p'-r,p') \otimes X''(p''-r,p'') \arrow[r, "\mu"] \arrow[d] & X(p'+p''-r,p'+p'') \arrow[d] \\
\Sigma \left(\begin{array}{c} X'(p'-2r,p'-r) \otimes X''(p''-r,p'') \\ \oplus \\ X'(p'-r,p') \otimes X''(p''-2r,p''-r) \end{array} \right) \arrow[r, "\Sigma(\mu+\mu)"] & \Sigma X(p'+p''-2r,p'+p''-r)
\end{tikzcd}.$$

As $\pi_* X(p-1,p) = E^1_p(X)$, the pairings obtained for $r=1$ give $E^1_{p'}(X') \mathbin{\overline{\otimes}} E^1_{p''}(X'') \to E^1_{p'+p''}(X)$. The above square, together with Lemma 2.21 identifying the left vertical composite and our conventions regarding Künneth maps, implies this satisfies the Leibniz rule, and hence passes to a pairing on E^r for all $r \geq 1$. By construction there are canonically commutative diagrams

$$\begin{tikzcd}
X'(p'-r,p') \otimes X''(p''-r,p'') \arrow[r] \arrow[d] & X(p'+p''-r,p'+p'') \arrow[d] \\
X'(p'-1,p') \otimes X''(p''-1,p'') \arrow[r] & X(p'+p''-1,p'+p'')
\end{tikzcd},$$

and these tell us that the pairing on E^r is induced from the pairing in $\mathcal{C}$. $\square$

We end with a remark concerning convergence of this product. Fix a pairing of towers $X' \otimes X'' \to X$. Suppose that $\mathcal{C}$ and $\mathcal{A}$ admit countable sums, and that these distribute across $\otimes$ and $\overline{\otimes}$. In particular, we obtain a pairing $X'(\infty) \otimes X''(\infty) \to X(\infty)$. Under the convergence conditions of Proposition 2.25, the pairing $E(X') \mathbin{\overline{\otimes}} E(X'') \to E(X)$ of Theorem 2.14 passes to $E^\infty_{p'}(X') \mathbin{\overline{\otimes}} E^\infty_{p''}(X'') \to E^\infty_{p'+p''}(X)$. As there are canonically commutative diagrams

$$\begin{tikzcd}
X'(p') \otimes X''(p'') \arrow[r] \arrow[d] & X(p'+p'') \arrow[d] \\
X'(\infty) \otimes X''(\infty) \arrow[r] & X(\infty)
\end{tikzcd},$$

this is the associated graded of the pairing $\pi_* X'(\infty) \mathbin{\overline{\otimes}} \pi_* X''(\infty) \to \pi_* X(\infty)$.

CHAPTER 3

Algebra

3.1. Algebraic theories

This section is primarily a compilation of a number of mostly classical definitions which are useful for understanding the algebraic structures we are interested in. In order to make this chapter mostly self-contained, we will recall in Subsection 3.1.1 the basic definitions and facts regarding algebraic theories needed from Chapter 2. In Subsection 3.1.2 and Subsection 3.1.3 we review the notions of bimodels and algebras over theories; in Subsection 3.1.4 we discuss how monoidal structures interact with theories; and in Subsection 3.1.5 we recall the notion of a distributive law. In Subsections 3.1.6–3.1.9 we cover the relevant notions of left-derived functors and Quillen cohomology, and describe how this plays out in the context of models for an algebra over a theory.

3.1.1. Review. We recall some definitions from Chapter 2, emphasizing the discrete case.

Definition 3.1 (Subsection 2.1.1).

(1) An *algebraic theory* is a category $\mathcal{P}$ which admits all small coproducts, and we say that $\mathcal{P}$ is a *discrete theory* if $\mathcal{P}$ is a 1-category.

(2) A theory $\mathcal{P}$ is *κ-bounded* for a regular cardinal κ if there exists a small full subcategory $\mathcal{P}_0 \subset \mathcal{P}$ closed under κ-small coproducts and satisfying the following κ-compactness condition: for every $P_0 \in \mathcal{P}_0$ and set of objects $\{P_i : i \in I\}$ of $\mathcal{P}$, the canonical map $\operatorname{colim}_{F \subset I, |F| < \kappa} \operatorname{Map}_{\mathcal{P}}(P_0, \coprod_{i \in F} P_i) \to \operatorname{Map}_{\mathcal{P}}(P_0, \coprod_{i \in I} P_i)$ is an equivalence. We say that $\mathcal{P}$ is *bounded* if $\mathcal{P}$ is κ-bounded for some κ.

(3) The category of *models* of an algebraic theory $\mathcal{P}$ is the full subcategory $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \subset \operatorname{Psh}(\mathcal{P})$ of small presheaves X on $\mathcal{P}$ such that for any set $\{P_i : i \in I\}$ of objects of $\mathcal{P}$, the canonical map $X(\coprod_{i \in I} P_i) \to \prod_{i \in I} X(P_i)$ is an equivalence. The category of *discrete models* of $\mathcal{P}$ is the full subcategory $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ of models whose underlying presheaf takes values in sets.

(4) An algebraic theory $\mathcal{P}$ is *Mal'cev* provided it satisfies a certain additional condition, equivalent when $\mathcal{P}$ is a discrete theory to the following: for every simplicial object $X \colon \Delta^{\mathrm{op}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ and every $P \in \mathcal{P}$, the simplicial set $X(P)$ is a Kan complex.

By *theory* we will always refer to a bounded Mal'cev theory. ◁

Throughout this chapter, $\mathcal{P}$ will always refer to some theory, possibly satisfying additional assumptions. We will abuse notation by implicitly identifying $\mathcal{P}$ as a full subcategory of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$. Except when giving some definitions and basic facts, $\mathcal{P}$ will be a discrete theory. We will recall some additional notation in Subsection 3.1.7 for models of $\mathcal{P}$ in the case where $\mathcal{P}$ is additive.

The structure of the category of models of $\mathcal{P}$ can be summarized as follows.

Lemma 3.1 (Section 2.1).

(1) $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is the free cocompletion of $\mathcal{P}$ under geometric realizations, and these are preserved by the embedding $\mathcal{M}\mathrm{odel}_{\mathcal{P}} \subset \mathrm{Psh}(\mathcal{P})$. In particular, if $X \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, then $\mathrm{Map}_{\mathcal{P}}(X, -)$ preserves geometric realizations if and only if X is a retract of some object of $\mathcal{P}$.
(2) Say $\mathcal{P}$ is κ-bounded, and fix $\mathcal{P}_0 \subset \mathcal{P}$ realizing this. Then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is equivalent to the category of presheaves on $\mathcal{P}_0$ which preserve κ-small coproducts. In particular, it is a κ-compactly generated presentable category.
(3) If $\mathcal{P}$ is discrete, then $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is the free 1-categorical cocompletion of $\mathcal{P}$ under reflexive coequalizers. In this case $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is the underlying ∞-category of Quillen's homotopy theory of simplicial objects in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, with localization realized by geometric realization. $\square$

We think of a theory $\mathcal{P}$ as encoding, and encoded by, natural operations on its models. This manifests as follows.

For $P \in \mathcal{P}$, write $\mathrm{ev}_P \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{G}\mathrm{pd}_{\infty}$ for the functor $\mathrm{ev}_P(X) = X(P)$.

Proposition 3.1. For $P, P' \in \mathcal{P}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{M}\mathrm{odel}_{\mathcal{P}},\, \mathcal{S}\mathrm{et})}(\pi_0 \mathrm{ev}_P, \pi_0 \mathrm{ev}_{P'}) \cong \pi_0 \mathrm{Map}_{\mathcal{P}}(P', P).$$

Proof. The Yoneda lemma gives a natural isomorphism

$$\mathrm{ev}_P(X) \simeq \mathrm{Map}_{\mathcal{P}}(P, X),$$

and thus

$$\pi_0 \mathrm{ev}_P(X) \cong \pi_0 \mathrm{Map}_{\mathcal{P}}(P, X) \cong \mathrm{Map}_{\mathrm{h}\mathcal{P}}(P, X).$$

In other words, $\pi_0 \mathrm{ev}_P$ is corepresented by P as a functor on the homotopy category of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$. We conclude with another application of the Yoneda lemma, yielding

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Fun}(\mathcal{M}\mathrm{odel}_{\mathcal{P}},\, \mathcal{S}\mathrm{et})}(\pi_0 \mathrm{ev}_P, \pi_0 \mathrm{ev}_{P'}) &\cong \mathrm{Hom}_{\mathrm{Fun}(\mathrm{h}\mathcal{M}\mathrm{odel}_{\mathcal{P}},\, \mathcal{S}\mathrm{et})}(\pi_0 \mathrm{ev}_P, \pi_0 \mathrm{ev}_{P'}) \\ &\cong \mathrm{Hom}_{\mathrm{h}\mathcal{P}}(P', P) \cong \pi_0 \mathrm{Map}_{\mathcal{P}}(P', P).\end{aligned} \qquad \square$$

We are interested in theories primarily as a tool for accessing their models.

Proposition 3.2.

(1) If $\mathcal{P}$ is an discrete additive theory, then $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ is a complete and cocomplete abelian category with enough projectives;
(2) If $\mathcal{A}$ is a cocomplete abelian category and $\mathcal{P} \subset \mathcal{A}$ is a full subcategory consisting of projective objects and closed under coproducts such that every $M \in \mathcal{A}$ is resolved by objects of $\mathcal{P}$, then $\mathcal{A} \simeq \mathrm{Model}_{\mathcal{P}}^{\heartsuit}$.

Proof. (1) Observe that as $\mathcal{P}$ is additive, every model $X\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ admits an essentially unique lift through $\mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{et}$. Thus $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ is equivalent to the category of $\mathcal{A}\mathrm{b}$-valued models of $\mathcal{P}$, which is a full subcategory of $\mathrm{Psh}(\mathcal{P}, \mathcal{A}\mathrm{b})$ closed under finite limits and colimits. This implies that $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ is abelian, and that it is complete and cocomplete with enough projectives follows from Lemma 3.1.

(2) Fix such $\mathcal{A}$ and $\mathcal{P} \subset \mathcal{A}$. As $\mathcal{A}$ admits small colimits, the restricted Yoneda embedding $h\colon \mathcal{A} \to \mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ admits a left adjoint $L\colon \mathrm{Model}_{\mathcal{P}}^{\heartsuit} \to \mathcal{A}$. As $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ is the free 1-categorical cocompletion of $\mathcal{P}$ under reflexive coequalizers, we must only verify that h is conservative, which follows from the assumption that every $M \in \mathcal{A}$ is resolved by objects of $\mathcal{P}$. $\square$

Example 3.1.

(1) If R is an ordinary associative ring and $\mathcal{R}$ is the category of free left R-modules, then $\mathcal{R}$ is a theory, $\mathrm{Model}_{\mathcal{R}}^{\heartsuit} \simeq \mathrm{LMod}_{R}^{\heartsuit}$ is equivalent to the category of ordinary left R-modules, and $\mathrm{Model}_{\mathcal{R}} \simeq \mathrm{LMod}_{R}^{\mathrm{cn}}$ is equivalent to the category of connective modules over the Eilenberg-MacLane spectrum HR.
(2) If G is a finite group and $\mathcal{B}_G$ is the Burnside category of finite G-sets, i.e. the additive completion of the category of finite G-sets and spans thereof, then $\mathcal{B}_G$ is a finitary theory and $\mathrm{Model}_{\mathcal{B}_G}^{\heartsuit}$ is equivalent to the category of G-Mackey functors.
(3) Let p be a prime and $\mathcal{P}$ be the category of p-completions of free abelian groups. Then $\mathcal{P}$ is an $\aleph_1$-bounded theory which is not ω-bounded, i.e. is not associated to a finitary theory. The category $\mathrm{Model}_{\mathcal{P}}^{\heartsuit}$ is equivalent to the category of Ext-p-complete abelian groups in the sense of [**BK72a**, Section VI.2.1], and $\mathrm{Model}_{\mathcal{P}}$ is equivalent to the category of connective $\mathbb{Z}$-modules which are p-complete in the sense of [**GM95**] or [**Lur18**, Chapter 7] (cf. Section 2.5). $\triangleleft$

Remark 3.1. At least up to Morita equivalence, finitary additive theories are equivalent to *ringoids*, i.e. small $\mathcal{A}\mathrm{b}$-enriched categories: if $\mathcal{C}$ is a finitary additive theory and $\mathcal{A} \subset \mathcal{C}$ is a subcategory generating $\mathcal{C}$ under finite sums and retracts, then $\mathrm{Model}_{\mathcal{C}}^{\heartsuit}$ is equivalent to the category of left $\mathcal{A}$-modules in the sense of [**Mit72**]. For example, if $\mathcal{C}$ is the theory of left modules over a ring R, then we may take $\mathcal{A} \subset \mathcal{C}$ to be the full subcategory on the single object R; here $\mathcal{A}$ is equivalent to R^{op} viewed as an $\mathcal{A}$b-enriched category with one object and $\mathrm{LMod}_R^{\heartsuit}$ is equivalent to the category of additive functors $R \to \mathcal{A}\mathrm{b}$. $\triangleleft$

Call a functor $U \colon \mathcal{D} \to \mathcal{C}$ *strongly monadic* if U preserves geometric realizations and is the forgetful functor of a monad adjunction. At least when $\mathcal{C}$ itself admits geometric realizations, it is equivalent to ask that $\mathcal{D} \simeq \mathcal{A}\mathrm{lg}_T$ for a monad T on $\mathcal{C}$ which preserves geometric realizations. The monads that we encounter will generally be of this form, as these are the monads which play well with theories.

LEMMA 3.2. If T is an accessible monad on $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ which preserves geometric realizations, and $T\mathcal{P} \subset \mathcal{A}\mathrm{lg}_T$ is the full subcategory spanned by the image of $\mathcal{P}$ under T, then $T\mathcal{P}$ is a theory and $\mathcal{M}\mathrm{odel}_{T\mathcal{P}} \simeq \mathcal{A}\mathrm{lg}_T$. Moreover, every conservative accessible functor $U \colon \mathcal{D} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ which preserves limits and geometric realizations arises this way.

PROOF. That $T\mathcal{P}$ is a theory is clear, and the equivalence $\mathcal{M}\mathrm{odel}_{T\mathcal{P}} \simeq \mathcal{A}\mathrm{lg}_T$ follows quickly from Lemma 3.1. The final statement follows from the crude monadicity theorem [**Lur17a**, Theorem 4.7.0.3]. $\square$

EXAMPLE 3.2. Let R be a commutative ring, and let $S\mathcal{R}$ the category of polynomial R-rings. This is the essential image of the theory of R-modules under the free functor $S \colon \mathcal{M}\mathrm{od}_R^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_R^{\heartsuit}$, so we can identify $\mathcal{M}\mathrm{odel}_{S\mathcal{R}}^{\heartsuit} \simeq \mathrm{C}\mathcal{R}\mathrm{ing}_R^{\heartsuit}$, and $\mathcal{M}\mathrm{odel}_{S\mathcal{R}} \simeq \mathrm{C}\mathcal{R}\mathrm{ing}_R$ is the homotopy theory of simplicial (commutative) R-rings. $\triangleleft$

3.1.2. Bimodels. Fix theories $\mathcal{P}$ and $\mathcal{P}'$.

LEMMA 3.3. The following concepts are equivalent:

(1) Models of $\mathcal{P}$ in $\mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\mathrm{op}}$;
(2) Left adjoint, or colimit-preserving, functors $H \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$, or equivalently, coproduct-preserving functors $H \colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$;
(3) Right adjoint, or limit-preserving accessible, or pointwise corepresentable, functors $H^{\vee} \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$.

PROOF. These follow directly from either Lemma 3.1 or the adjoint functor theorems for presentable categories [**Lur17a**, Proposition 5.5.2.2, Corollary 5.5.2.9]. In addition, we can make the corepresentability condition of (3) explicit: $H^{\vee}(M)(P) \simeq \mathrm{Map}_{\mathcal{P}}(H(P), M)$. $\square$

We call the concept encoded in Lemma 3.3 that of a $\mathcal{P}$-$\mathcal{P}'$-*bimodel*; when $\mathcal{P} = \mathcal{P}'$, we will just call these $\mathcal{P}$-bimodels. We refer the reader to Wraith [**Wra70**] and Freyd [**Fre66**] for classical treatments of bimodels, as well as of algebras, defined below. We will consistently adhere to the convention that by bimodel we refer to the underlying left adjoint H, and that in this case $H^{\vee}$ is written for its right adjoint. For $P \in \mathcal{P}$ and $P' \in \mathcal{P}'$, we may at times write $H_{P,P'} = H(P)(P')$ and $H^\vee_{P,P'} = H^\vee(P)(P')$; note that these are covariant in the first variable and contravariant in the second.

Example 3.3. Let $\mathcal{P}$ be the theory of groups, R a commutative ring, and $\mathcal{P}'$ the theory of commutative R-rings. To a commutative Hopf algebra H over R, one may associate the functor

$$\mathrm{C}\mathcal{R}\mathrm{ing}_R(H,-)\colon \mathrm{C}\mathcal{R}\mathrm{ing}_R \to \mathcal{G}\mathrm{rp}.$$

This is a $\mathcal{P}'$-$\mathcal{P}$-bimodel, and all discrete $\mathcal{P}'$-$\mathcal{P}$-bimodels arise this way. ◁

Call a $\mathcal{P}'$-$\mathcal{P}$-bimodel H *projective* if $H(P)$ is projective for all $P \in \mathcal{P}$; equivalently, if H restricts to a functor $H\colon \mathcal{P} \to \mathcal{P}'$, at least up to idempotent completion of $\mathcal{P}'$.

If $\mathcal{P}''$ is another theory and H' is a $\mathcal{P}''$-$\mathcal{P}'$-bimodel, then we can compose to obtain the $\mathcal{P}''$-$\mathcal{P}$-bimodel $H' \circ H$. This has right adjoint $(H' \circ H)^\vee \simeq H^\vee \circ H'^\vee$.

Remark 3.2. Although we are primarily interested in discrete bimodels, we are still interested in derived aspects of these. Even supposing that the theories in question are discrete, there are two possible ambiguities that arise:

(1) Coproduct-preserving functors $H\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}^\heartsuit_{\mathcal{P}'}$ are not equivalent to coproduct-preserving functors $H\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$ such that $H(P)$ is discrete for all $P \in \mathcal{P}$;
(2) Even if $H\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$ and $H'\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}''}$ deserve to be called discrete bimodels, the same need not hold for the composite $H' \circ H$.

The second ambiguity is not major, being no different than the ambiguity between a derived tensor product and a non-derived tensor product. The first ambiguity is more subtle, amounting to the observation that discrete models need not be closed under coproducts in the category of all models. When $\mathcal{P}$ is additive, this amounts to the observation that infinite sums need not be exact in a general abelian category with enough projectives.

Neither of these will be major issues for us. In practice, where they might otherwise cause problems, we will simply assume that our bimodels are projective, at which point both of these ambiguities vanish. However, for the sake of avoiding projectivity assumptions where they are not relevant, we take the following convention. When we are dealing with the purely discrete aspects of bimodels, we take as our discrete bimodels those which correspond to coproduct-preserving functors $H\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}^\heartsuit_{\mathcal{P}'}$. When we are dealing with homotopical aspects of bimodels, we take as our discrete bimodels those which correspond to coproduct-preserving functors $H\colon \mathcal{P} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}'}$ that land in $\mathcal{M}\mathrm{odel}^\heartsuit_{\mathcal{P}'}$. ◁

Example 3.4. Let A and B be ordinary associative algebras with theories $\mathcal{A}$ and $\mathcal{B}$ of left modules. Then discrete $\mathcal{B}$-$\mathcal{A}$-bimodels are equivalent to B-A-bimodules. It is worth spelling out some aspects of this example explicitly to indicate the conventions that arise for working with bimodules. We only consider discrete bimodels in this example, although similar observations hold in the derived setting (where general $\mathcal{B}$-$\mathcal{A}$-bimodels are equivalent to connective modules over the ring spectrum $B \otimes_{\mathbb{S}} A^{\mathrm{op}}$). To a discrete B-A-bimodule H, one can associate the bimodel

$$\begin{aligned} H\colon \mathrm{L}\mathcal{M}\mathrm{od}_A^{\heartsuit} &\to \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}, \qquad & H(M) &= H \otimes_A M; \\ H^\vee\colon \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit} &\to \mathrm{L}\mathcal{M}\mathrm{od}_A^{\heartsuit}, \qquad & H^\vee(M) &= \mathrm{Hom}_B(H, M). \end{aligned}$$

Here, B acts on $H \otimes_A M$ by

$$b \cdot (h \otimes m) = (bh) \otimes m,$$

and A acts on $\mathrm{Hom}_B(H, M)$ by

$$(a \cdot f)(h) = f(ha).$$

The bimodel H is projective precisely when the bimodule H is projective as a left B-module. The dual functor $H^\vee$ encodes more information than the ordinary dual bimodule $\mathrm{Hom}_B(H, B)$, and the latter can be recovered from the former by considering the restriction of $H^\vee$ to the category of finitely generated free B-modules. On the other hand, if H is projective, then we can equip $\mathrm{Hom}_B(H, B)$ with a natural topology as a right B-module such that $H^\vee(M) \simeq \mathrm{Hom}_B(H, B) \mathbin{\widehat{\otimes}}_B M$, where B acts on $\mathrm{Hom}_B(H, B)$ on the right by $(f \cdot b)(h) = f(bh)$.

If C is another ordinary associative algebra, $\mathcal{C}$ is its theory of left modules, and H' is a discrete $\mathcal{C}$-$\mathcal{B}$-bimodel, then under the correspondence between bimodules and bimodels we identify

$$H' \circ H \simeq H' \otimes_B H.$$

The isomorphism $(H' \circ H)^\vee \cong H^\vee \circ H'^\vee$ is given by the maps

$$\begin{gathered} \theta\colon \ \mathrm{Hom}_B(H, \mathrm{Hom}_C(H', M)) \to \mathrm{Hom}_C(H' \otimes_B H, M), \\ \theta(f)(h' \otimes h) = f(h)(h'). \end{gathered}$$

Taking $A = B = C$, this is an enhancement of the duality pairing

$$\begin{gathered} \theta\colon \ \mathrm{Hom}_A(H, A) \otimes_A \mathrm{Hom}_A(H', A) \to \mathrm{Hom}_A(H' \otimes_A H, A), \\ \theta(f \otimes f')(h' \otimes h) = f'(h' f(h)) \end{gathered}$$

of bimodules. $\triangleleft$

3.1.3. Algebras.

Definition 3.2. a $\mathcal{P}$-*algebra* consists of a $\mathcal{P}$-bimodel F together with the additional structure of a monad on F, or equivalently, of a comonad on $F^\vee$. An F-*model* is an algebra for the monad F, or equivalently, coalgebra for the comonad $F^\vee$. $\triangleleft$

If F is a $\mathcal{P}$-algebra, then $\mathcal{A}\mathrm{lg}_F \simeq \mathcal{M}\mathrm{odel}_{F\mathcal{P}}$ by Lemma 3.2; we will abbreviate this to $\mathcal{M}\mathrm{odel}_F$. The forgetful functor $\mathcal{M}\mathrm{odel}_F \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ is *plethystic*: it is both monadic and comonadic. Conversely, every category plethystic over $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ arises from a $\mathcal{P}$-algebra. Heuristically, $\mathcal{P}$-algebras are those theories that can be obtained from $\mathcal{P}$ by adjoining sufficiently well-behaved unary operations and relations.

Example 3.5. The following are examples of discrete algebras.

(1) Let R be an ordinary associative algebra and $\mathcal{R}$ be the theory of left R-modules. Then a discrete $\mathcal{R}$-algebra is equivalent to a discrete R-bimodule A equipped with the structure of a monoid in the category of R-bimodules. Thus discrete $\mathcal{R}$-algebras are equivalent to ordinary associative algebras equipped with an algebra map from R; we will just call these *ordinary R-algebras*. In particular, even when R is commutative, it need not be central in its algebras.
(2) Let R be a commutative ring and $S\mathcal{R}$ be the the category of polynomial R-algebras, as in Example 3.2. Then discrete $S\mathcal{R}$-algebras were studied by Tall-Wraith [**TW70**] under the name of biring triples, and more recently by Borger-Wieland [**BW05**] under the name of R-plethories. Our main examples of algebras over nonadditive theories are essentially of this form. We will study the relevant context in Section 3.3, allowing for bases more general than just commutative rings.
(3) Let G be a finite group, $\mathcal{B}_G$ be the Burnside category of finite G-sets, as in Example 3.1, and $S\mathcal{B}_G$ be the category of commutative green functors free on objects of $\mathcal{B}_G$, so that $\mathcal{M}\mathrm{odel}_{S\mathcal{B}_G}^{\heartsuit}$ is the category of commutative green functors. Let $\mathcal{T}\mathrm{amb}_G^{\heartsuit}$ be the category of G-Tambara functors. Then $\mathcal{T}\mathrm{amb}_G^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{S\mathcal{B}_G}^{\heartsuit}$ preserves limits and colimits, and so realizes $\mathcal{T}\mathrm{amb}_G^{\heartsuit}$ as the category of models for an $S\mathcal{B}_G$-algebra. Thus G-tambara functors are $\mathcal{B}_G$-plethories in the sense that we will study in Section 3.3, although we do not know whether they satisfy the various niceness properties introduced there. See [**BH19**] for more on this context. $\triangleleft$

Remark 3.3. Plethystic functors also arise in more homotopical contexts.

(1) Let R be a commutative ring, $\mathcal{R}$ be the theory of R-modules, and $\mathbb{P}\mathcal{R}$ be the category of $\mathbb{E}_\infty$ algebras over R which are free on a discrete free R-module, i.e. of the form $\mathbb{P}R^{\oplus I}$ where $\mathbb{P}\colon \mathcal{M}\mathrm{od}_R \to \mathcal{C}\mathrm{Alg}_R$ is the free $\mathbb{E}_\infty$ algebra functor. Then $\mathcal{M}\mathrm{odel}_{\mathbb{P}R} \simeq \mathcal{C}\mathrm{Alg}_R^{\mathrm{cn}}$ is equivalent to the category of connective $\mathbb{E}_\infty$ algebras over R. The homotopy category $\mathrm{h}(\mathbb{P}\mathcal{R}) \simeq S\mathcal{R}$ is equivalent to the category of polynomial R-rings, and restriction along the truncation $\mathbb{P}\mathcal{R} \to S\mathcal{R}$ gives a forgetful functor $U\colon \mathrm{C}\mathcal{R}\mathrm{ing}_R \to \mathcal{C}\mathrm{Alg}_R^{\mathrm{cn}}$. The functor U automatically preserves limits and geometric realizations, and it preserves coproducts as these are given by $\otimes_R$ in either category. Thus U is plethystic, and realizes $S\mathcal{R}$ as a

$\mathbb{P}\mathcal{R}$-algebra. We refer the reader to [**Lur18**, Chapter 25] for a more detailed discussion of the relation between $\mathrm{C}\mathcal{R}\mathrm{ing}_R$ and $\mathcal{C}\mathrm{Alg}_R^{\mathrm{cn}}$. We will briefly revisit this example in Example 4.4.

(2) Let G be a finite group and $\mathcal{O}' \subset \mathcal{O}$ be G-coefficient systems in the sense of [**BH15**]. Then the forgetful functor $\mathcal{A}\mathrm{lg}_{\mathcal{O}} \to \mathcal{A}\mathrm{lg}_{\mathcal{O}'}$ is plethystic, where $\mathcal{A}\mathrm{lg}_{\mathcal{O}}$ is the category of algebras over the N_∞-operad associated to $\mathcal{O}$.

These examples point to a possible theory of "spectral plethories" encoding various refinements of the basic notion of commutative multiplication encoded by the $\mathbb{E}_\infty$ operad. $\triangleleft$

3.1.4. Monoidal products. Suppose that $\mathcal{M}\mathrm{odel}_{\mathcal{P}}$ has been equipped with some form of monoidal product $\otimes$ which preserves colimits in each variable. If moreover the monoidal product preserves the full subcategory $\mathcal{P} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, then it is determined by its restriction to $\mathcal{P}$, from which it can be recovered by Day convolution. In this case, one might call $\mathcal{P}$ a *monoidal theory*. In this case, if $X', X'' \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$, then $X' \otimes X''$ can be identified as the left Kan extension of the functor $(P', P'') \mapsto X'(P') \times X''(P'')$ along the product $\otimes \colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$.

Note in particular the following: if $\mathcal{P}$ is a monoidal theory, then for any $P', P'' \in \mathcal{P}$, there is a natural pairing

$$\mathrm{ev}_{P'} \times \mathrm{ev}_{P''} \to \mathrm{ev}_{P' \otimes P''}$$

satisfying all the coherences one would expect coming from $\otimes$. This is an advantage of working with algebraic theories without specified sorts, as the presence of automorphisms of objects of $\mathcal{P}$ has not been hidden.

3.1.5. Compositions. If k is an ordinary commutative ring, and A and B are ordinary k-algebras in which k is central, then the tensor product $A \otimes_k B$ naturally carries the structure of a k-algebra, with product

$$m \otimes m \circ A \otimes \tau \otimes B \colon A \otimes_k B \otimes_k A \otimes_k B \cong A \otimes_k A \otimes_k B \otimes_k B \to A \otimes_k B.$$

This is not true for general k-algebras, or for k noncommutative: we have relied on centrality in order to use the switch map $\tau \colon A \otimes_k B \simeq B \otimes_k A$. Axiomatizing this leads to the notion of a distributive law, discovered by Beck [**Bec69**]. We summarize the relevant definitions here.

Definition 3.3. Let $\mathcal{C}$ be a 1-category and F and T be monads on $\mathcal{C}$.

(1) A *composition* of T with F is the structure of a monad on the composite functor TF satisfying the following conditions:
   (a) Both $T\eta_F \colon T \to TF$ and $\eta_T F \colon F \to TF$ are maps of monads;
   (b) The composite

$$m_{TF} \circ T\eta_F \eta_T F \colon TF \to TFTF \to TF$$

is the identity.

(2) A *distributive law* of F across T is a natural transformation $c\colon FT \to TF$ such that the diagrams

$$\begin{tikzcd} & T \arrow[dl, "\eta_F T"'] \arrow[dr, "T\eta_F"] & \\ FT \arrow[rr, "c"] & & TF \end{tikzcd} \qquad \begin{tikzcd} & F \arrow[dl, "F\eta_T"'] \arrow[dr, "\eta_T F"] & \\ FT \arrow[rr, "c"] & & TF \end{tikzcd}$$

$$\begin{tikzcd} FTT \arrow[r, "cT"] \arrow[d, "Fm_T"] & TFT \arrow[r, "Tc"] & TTF \arrow[d, "m_T F"] \\ FT \arrow[rr, "c"] & & TF \end{tikzcd} \qquad \begin{tikzcd} FFT \arrow[r, "Fc"] \arrow[d, "m_F T"] & FTF \arrow[r, "cF"] & TFF \arrow[d, "Tm_F"] \\ FT \arrow[rr, "c"] & & TF \end{tikzcd}$$

commute.

(3) A *distributive square* is a diagram of categories

$$\begin{tikzcd} \mathcal{D}' \arrow[r, "V'"] \arrow[d, shift left, "U'"] & \mathcal{D} \arrow[d, shift left, "U"] \\ \mathcal{C}' \arrow[r, "V"] \arrow[u, shift left, "T'"] & \mathcal{C} \arrow[u, shift left, "T"] \end{tikzcd}$$

such that

(a) The diagram commutes with T and T' omitted;

(b) The pairs $T' \dashv U'$ and $T \dashv U$ are adjoint;

(c) The mate $TV \to V'T'$ is an isomorphism.

(4) A *monadic distributive square* is a distributive square as above such that moreover

(a) There are further left adjoints $F' \dashv V'$ and $F \dashv V$;

(b) Each of these adjunctions are monadic adjunctions. ◁

We extend the definitions of distributive squares and monadic distributive squares to allow for the categories involved to not necessarily be 1-categories; these are essentially the left adjointable squares of [**Lur17a**, Definition 4.7.4.13]. Heuristically, a monadic distributive square is a square

$$\begin{tikzcd} \mathcal{D}' \arrow[r, shift left, "V'"] \arrow[d, shift left, "U'"] & \mathcal{D} \arrow[l, shift left, "F'"] \arrow[d, shift left, "U"] \\ \mathcal{C}' \arrow[r, shift left, "V"] \arrow[u, shift left, "T'"] & \mathcal{C} \arrow[l, shift left, "F"] \arrow[u, shift left, "T"] \end{tikzcd}$$

of monadic adjunctions such that "$T = T'$". This is of course dependent on the orientation of the square.

Lemma 3.4. Let $\mathcal{C}$ be a 1-category and F and T be monads on $\mathcal{C}$. The following concepts are equivalent:

(1) Compositions of T with F;

(2) Distributive laws of F across T;

(3) Monadic distributive squares of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[r, shift left, "V'"] \arrow[d, shift left, "U'"] & \mathcal{A}\mathrm{lg}_T \arrow[l, shift left, "F'"] \arrow[d, shift left, "U"] \\
\mathcal{A}\mathrm{lg}_F \arrow[u, shift left, "T'"] \arrow[r, shift left, "V"] & \mathcal{C} \arrow[l, shift left, "F"] \arrow[u, shift left, "T"]
\end{tikzcd}$$

Proof. The equivalence of these notions is proved in [**Bec69**]; we just recall here the method of translation between the three. Given a composition of T with F, we obtain a distributive law of F across T by the composite

$$c = m_{TF} \circ \eta_T FT\eta_F \colon FT \to TFTF \to TF.$$

Conversely, given a distributive law $c\colon FT \to TF$, we can construct a composition of T with F via

$$\eta_{TF} = \eta_T\eta_F \colon I \to TF,$$
$$m_{TF} = m_T m_F \circ TcF \colon TFTF \to TTFF \to TF.$$

Given a composition of T with F, we obtain a diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{lg}_{TF} \arrow[r, shift left, "V'"] \arrow[d, shift left, "U'"] & \mathcal{A}\mathrm{lg}_T \arrow[l, shift left, "F'"] \arrow[d, shift left, "U"] \\
\mathcal{A}\mathrm{lg}_F \arrow[u, shift left, "T'"] \arrow[r, shift left, "V"] & \mathcal{C} \arrow[l, shift left, "F"] \arrow[u, shift left, "T"]
\end{tikzcd}$$

of monadic functors. We claim this is distributive, i.e. $TV \cong V'T'$ where $T' \dashv U'$. Indeed, it is sufficient to verify this on free F-algebras and after forgetting to $\mathcal{C}$, where this is just the identification $UTVF \cong UV'T'F$. Conversely, given a monadic distributive square as in the statement of the lemma, we obtain a distributive law of F across T by the composite

$$FT = VFUT \to VU'F'T \cong UTVF = TF,$$

the arrow being obtained from the mate $FU \to U'F'$. □

Example 3.6. Let k be an ordinary associative algebra and A and B be ordinary k-algebras. Then

(1) Compositions of A with B are algebra structures on $A \otimes_k B$ such that
   (a) $(a' \otimes 1) \cdot (a'' \otimes 1) = a'a'' \otimes 1$ and $(1 \otimes b') \cdot (1 \otimes b'') = 1 \otimes b'b''$;
   (b) $(a \otimes 1) \cdot (1 \otimes b) = a \otimes b$.
(2) Distributive laws of B across A are maps $c\colon B \otimes_k A \to A \otimes_k B$ of k-bimodules such that
   (a) $c(1 \otimes a) = a \otimes 1$ and $c(b \otimes 1) = 1 \otimes b$;

(b) If we write $c(b \otimes a) = \sum a_{(i)} \otimes b_{(i)}$ for a placeholder symbol i, then $\sum a'_{(1)} \otimes a''_{(2)} \otimes (b_{(1)})_{(2)} = \sum (a' \otimes a'')_{(3)} \otimes b_{(3)}$ and $\sum (a_{(1)})_{(2)} \otimes b'_{(2)} \otimes b''_{(1)} = \sum a_{(3)} \otimes (b'b'')_{(3)}$.

(3) Given a commutative diagram

$$\begin{tikzcd} C & A \arrow[l] \\ B \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

of algebra maps, there is an associated commutative diagram

$$\begin{tikzcd} \mathrm{L}\mathcal{M}\mathrm{od}_C^\heartsuit \arrow[r] \arrow[d] & \mathrm{L}\mathcal{M}\mathrm{od}_A^\heartsuit \arrow[d] \\ \mathrm{L}\mathcal{M}\mathrm{od}_B^\heartsuit \arrow[r] & \mathrm{L}\mathcal{M}\mathrm{od}_k^\heartsuit \end{tikzcd}$$

of monadic forgetful functors. The mate of this diagram is given by maps $A \otimes_k M \to C \otimes_B M$ defined for left B-modules M, and is a natural isomorphism when it evaluates on B to an isomorphism $A \otimes_k B \cong C$.

Even when each of k, A, and B are commutative, these notions do not collapse. For example, if $\mathbb{H}$ is the ring of quaternions, then

$$\begin{tikzcd} \mathbb{H} & \mathbb{C} \arrow[l, "f"'] \\ \mathbb{C} \arrow[u, "g"] & \mathbb{R} \arrow[l] \arrow[u] \end{tikzcd}$$

satisfies the conditions of (3), where $f(i) = j$ and $g(i) = k$. The distributive law is the map $c\colon \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ given by $c(i \otimes i) = -i \otimes i$, and otherwise by the standard symmetry. $\triangleleft$

For the most part, we will encounter distributive laws in the form of monadic distributive squares, and the theory of distributive laws then provides a method for understanding the categories involved. In general, this is an instance of the following philosophy: it is often easier to construct the category of algebras over a monad than it is to construct the monad itself. The theory of distributive laws gives a way of accessing the monad associated to categories constructed by such indirect methods; the following are some typical examples.

Example 3.7. Let k be an ordinary commutative ring, B be an ordinary k-bialgebra, and A be a monoid in the monoidal category $(\mathrm{L}\mathcal{M}\mathrm{od}_B^\heartsuit, \otimes_k)$, with resulting category $\mathrm{L}\mathcal{M}\mathrm{od}_A(\mathrm{L}\mathcal{M}\mathrm{od}_B^\heartsuit)$ of modules therein. Then

$$\begin{tikzcd} \mathrm{L}\mathcal{M}\mathrm{od}_A(\mathrm{L}\mathcal{M}\mathrm{od}_B^\heartsuit) \arrow[r] \arrow[d] & \mathrm{L}\mathcal{M}\mathrm{od}_A^\heartsuit \arrow[d] \\ \mathrm{L}\mathcal{M}\mathrm{od}_B^\heartsuit \arrow[r] & \mathcal{M}\mathrm{od}_k^\heartsuit \end{tikzcd}$$

is a monadic distributive square, and thus $\mathrm{L}\mathcal{M}\mathrm{od}_A(\mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{A\otimes_k B}^{\heartsuit}$ for some k-algebra structure on $A \otimes_k B$. Using Lemma 3.4, one can compute that this k-algebra structure is the "semi-tensor product" [**MP65**], given by

$$(a' \otimes b') \cdot (a'' \otimes b'') = \sum a'(b'_{(1)} \cdot a'') \otimes b'_{(2)} b'',$$

where we have written $\Delta(b) = \sum b_{(1)} \otimes b_{(2)}$ for the coproduct on B. Equivalently, the distributive law is given by $c(b \otimes a) = \sum (b_{(1)} \cdot a) \otimes b_{(2)}$. $\triangleleft$

Example 3.8. Let $\mathcal{P}$ be a discrete theory, F a discrete $\mathcal{P}$-algebra, and $B \in \mathcal{M}\mathrm{odel}_F^{\heartsuit}$. Then

$$\begin{tikzcd} B/\mathcal{M}\mathrm{odel}_F^{\heartsuit} \arrow[r] \arrow[d] & B/\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \arrow[d] \\ \mathcal{M}\mathrm{odel}_F^{\heartsuit} \arrow[r] & \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \end{tikzcd}$$

is a monadic distributive square. The distributive law is just the map

$$F(B \textstyle\coprod -) \simeq F(B) \coprod F(-) \to B \coprod F(-)$$

obtained from the fact that F preserves coproducts and the F-model structure of B. $\triangleleft$

See Example 3.19 for an explicit instance of the preceding examples.

3.1.6. Left-derived functors. Fix two discrete theories $\mathcal{P}$ and $\mathcal{P}'$.

Definition 3.4. Fix an arbitrary functor $F\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, and let f denote the composite

$$f\colon \mathcal{P}' \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \subset \mathcal{M}\mathrm{odel}_{\mathcal{P}}.$$

The *total left-derived functor* of F is the functor

$$\mathbb{L}F = f_! \colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}$$

obtained from f by left Kan extension. When $\mathcal{P}$ is pointed, we abbreviate $\mathbb{L}_n F = \pi_n \mathbb{L}F$. $\triangleleft$

Total left-derived functors can be computed in the usual way, by taking projective resolutions (Proposition 2.4). Their identification with a left Kan extension is a situation where the use of infinitary theories simplifies the story.

Example 3.9. Let $F\colon \mathcal{M}\mathrm{od}_{\mathbb{Z}_p}^{\heartsuit} \to \mathcal{M}\mathrm{od}_{\mathbb{Z}_p}^{\heartsuit}$ denote the functor of p-adic completion. Then F is neither left nor right exact in general. Nonetheless, we may consider the total left-derived functor $\mathbb{L}F$. This has the following properties:

(1) $\mathbb{L}F$ gives the correct notion of p-completion for the homotopy theory $\mathcal{M}\mathrm{od}_{\mathbb{Z}_p}^{\mathrm{cn}}$;
(2) If $M \in \mathcal{M}\mathrm{od}_{\mathbb{Z}_p}^{\heartsuit}$, then $\mathbb{L}FM$ is 1-truncated, $\mathbb{L}_0 FM$ is the Ext-p-completion of M, and $\mathbb{L}_1 F$ is the Hom-p-completion of M in the sense of [**BK72a**, Section VI.2.1].

This is a purely infinitary construction, as F restricts to the identity on the category of finitely generated $\mathbb{Z}_p$-modules. $\triangleleft$

3.1.7. Unbounded derived categories. If $\mathcal{P}$ is an additive theory, then we will write

$$\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit} = \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}, \qquad \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} = \mathcal{M}\mathrm{odel}_{\mathcal{P}},$$

and further define $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ to be the category of $\mathcal{S}\mathrm{p}$-valued models of $\mathcal{P}$. There are then fully faithful embeddings $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \subset \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ is the stabilization of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}}$ (cf. Section 2.3).

In particular, for $X, Y \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, there is a mapping spectrum $\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X, Y)$ with

$$\Omega^{\infty-n}\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X, Y) \simeq \mathrm{Map}_{\mathcal{P}}(X, \Sigma^n Y),$$

and we write

$$\mathrm{Ext}_{\mathcal{P}}^n(X, Y) = \pi_{-n}\mathcal{E}\mathrm{xt}_{\mathcal{P}}(X, Y).$$

When $\mathcal{P}$, X and Y are discrete, these are the usual Ext groups defined for the abelian category $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$.

3.1.8. Quillen cohomology. Fix a discrete theory $\mathcal{P}$. Then $\mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit})$ is equivalent to the category of $\mathcal{A}\mathrm{b}$-valued models of $\mathcal{P}$, and this category is strongly monadic over $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. Write the left adjoint as $D\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit} \to \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit})$, so that $\mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{D\mathcal{P}}^{\heartsuit}$. Here, $D\mathcal{P}$ is an additive theory, so the notation of Subsection 3.1.7 applies.

Definition 3.5. For $A \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}$ and $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{D\mathcal{P}}$, define

$$\mathcal{H}_{\mathcal{P}}(A; M) = \mathcal{E}\mathrm{xt}_{D\mathcal{P}}(\mathbb{L}DA, M), \qquad H_{\mathcal{P}}^n(A; M) = \pi_{-n}\mathcal{H}_{\mathcal{P}}(A; M) = \mathrm{Ext}_{D\mathcal{P}}^n(\mathbb{L}DA, M).$$

Equivalently,

$$\mathcal{H}_{\mathcal{P}}^n(A; M) = \Omega^{\infty-n}\mathcal{H}_{\mathcal{P}}(A; M) = \mathrm{Map}_{\mathcal{P}}(A, B^n M), \qquad H_{\mathcal{P}}^n(A; M) = \pi_0\mathcal{H}_{\mathcal{P}}^n(A; M).$$

These are the *Quillen cohomology of A with coefficients in M.* $\triangleleft$

Often the theory at hand is instead of the form $\mathcal{P}/B$ for some theory $\mathcal{P}$ and $B \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, as in this case $\mathcal{M}\mathrm{odel}_{\mathcal{P}/B} \simeq \mathcal{M}\mathrm{odel}_{\mathcal{P}}/B$. Write D_B for the relevant functor of abelianization. Call $B \in \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ *smooth* if $\mathbb{L}D_B B$ is discrete and projective. When $\mathcal{P}$ is the theory of R-rings for some commutative ring R, this is not quite the standard notion of smoothness, as we have imposed no finiteness conditions.

Lemma 3.5. Given $f\colon B \to C$, there is an equivalence $\mathbb{L}D_C B \simeq f_!\mathbb{L}D_B B$, where $f_!$ is the total derived functor of the left adjoint to pullback $f^*\colon \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/C) \to \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/B)$. In particular, if B is smooth, then $\mathbb{L}D_C B$ is discrete and projective for any map f.

Proof. Observe that the diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/C) \arrow[r, "f^*"] \arrow[d] & \mathcal{A}\mathrm{b}(\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/B) \arrow[d] \\
\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/C \arrow[r, "f^*"] & \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}/B
\end{tikzcd}$$

of right adjoints commutes, and continues to commute upon passage to derived functors. The lemma follows upon taking left adjoints. □

3.1.9. Cohomology over an algebra. Fix a discrete theory $\mathcal{P}$ and $\mathcal{P}$-algebra F. We would like to be able to compute the Quillen cohomology of F-models.

Lemma 3.6. Suppose $U\colon \mathcal{M}\mathrm{odel}_{\mathcal{P}'}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is strongly monadic. Then the induced map $V\colon \mathcal{M}\mathrm{odel}_{D\mathcal{P}'}^{\heartsuit} \to \mathcal{M}\mathrm{odel}_{D\mathcal{P}}^{\heartsuit}$ is strongly monadic, and is plethystic whenever U is.

Proof. It is easily seen that V is strongly monadic. As V is additive, to be plethystic it is sufficient for V to preserve filtered colimits, for which it is sufficient that U preserves filtered colimits, which holds if U is plethystic. □

What makes an algebra F special is the existence of the limit-preserving comonad $F^{\vee}$. Heuristically, this is because $F^{\vee}$ preserves algebraic structure. In particular we can identify abelian group objects in $\mathcal{M}\mathrm{odel}_{F}^{\heartsuit} \simeq \mathrm{Co}\mathcal{A}\mathrm{lg}_{F^{\vee}}^{\heartsuit}$ using the following.

Proposition 3.3. Let $\mathcal{C}$ be a 1-category with finite products, and G be a comonad on $\mathcal{C}$ which preserves these. Then

(1) $\mathrm{Co}\mathcal{A}\mathrm{lg}_G \to \mathcal{C}$ creates finite products;
(2) The resulting forgetful functor $\mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G) \to \mathcal{A}\mathrm{b}(\mathcal{C})$ is comonadic;
(3) The diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G) \arrow[r] \arrow[d, "U'"] & \mathcal{A}\mathrm{b}(\mathcal{C}) \arrow[d, "U"] \\
\mathrm{Co}\mathcal{A}\mathrm{lg}_G \arrow[r] & \mathcal{C}
\end{tikzcd}$$

of forgetful functors is Cartesian whenever U is fully faithful;
(4) The natural transformation $U' \circ G' \to G \circ U$ fitting in the diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G) \arrow[d, "U'"] & \mathcal{A}\mathrm{b}(\mathcal{C}) \arrow[l, "G'"'] \arrow[d, "U"] \\
\mathrm{Co}\mathcal{A}\mathrm{lg}_G & \mathcal{C} \arrow[l, "G"']
\end{tikzcd}$$

is an isomorphism;
(5) If U admits a left adjoint D, then D lifts to a left adjoint $D'\colon \mathrm{Co}\mathcal{A}\mathrm{lg}_G \to \mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G)$ making the diagram in (3) distributive.

Proof. (1) This is clear.

(2, 4) As G preserves finite products, it lifts to a comonad G' on $\mathcal{A}\mathrm{b}(\mathcal{C})$. A G'-coalgebra consists of some $A \in \mathcal{A}\mathrm{b}(\mathcal{C})$ together with a coaction $A \to GA$ which is a map of abelian group objects, i.e. such that the diagram

$$\begin{tikzcd} A \times A \arrow[r] \arrow[d] & GA \times GA \arrow[d] \\ A \arrow[r] & GA \end{tikzcd}$$

commutes. Looking at it a different way, this is the same as asking for A to be an abelian group object in $\mathrm{Co}\mathcal{A}\mathrm{lg}_G$, so $\mathrm{Co}\mathcal{A}\mathrm{lg}_{G'} \simeq \mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G)$.

(3) If $\mathcal{A}\mathrm{b}(\mathcal{C}) \to \mathcal{C}$ is fully faithful, then the above diagram automatically commutes for any choice of multiplication $A\times A \to A$ and coaction $A \to GA$, and the claim quickly follows.

(5) Given the left adjoint D, there is a left adjoint $D' \colon \mathrm{Co}\mathcal{A}\mathrm{lg}_G \to \mathcal{A}\mathrm{b}(\mathrm{Co}\mathcal{A}\mathrm{lg}_G)$ sending a G-coalgebra $A \to GA$ to the G'-coalgebra with coaction the unique dashed arrow filling in

$$\begin{tikzcd} A \arrow[r] \arrow[d] & GA \arrow[d] \\ DA \arrow[r, dashed] & GDA \end{tikzcd}$$

as a map of abelian group objects in $\mathcal{C}$. This has the desired properties. $\square$

By Proposition 3.3, if $B \in \mathcal{M}\mathrm{odel}_F^\heartsuit$ and we are treating B as an model of $\mathcal{P}$ equipped with extra structure, then the notation $D(B)$ is unambiguous, for the abelianization of B is the same when computed in $\mathcal{M}\mathrm{odel}_F^\heartsuit$ or $\mathcal{M}\mathrm{odel}_\mathcal{P}^\heartsuit$. However, the notation $\mathbb{L}D(B)$ is ambiguous; thus for the moment we write $\mathbb{L}D'$ for the derived abelianization of F-models.

Call the algebra F *smooth* if $F(P)$ is smooth for all $P \in \mathcal{P}$.

Proposition 3.4. If F is smooth, then the diagram

$$\begin{tikzcd} \mathrm{L}\mathcal{M}\mathrm{od}_{DF}^{\mathrm{cn}} \arrow[r, "V'"] \arrow[d] & \mathrm{L}\mathcal{M}\mathrm{od}_{D\mathcal{P}}^{\mathrm{cn}} \arrow[d] \\ \mathcal{M}\mathrm{odel}_F \arrow[r, "V"] & \mathcal{M}\mathrm{odel}_\mathcal{P} \end{tikzcd}$$

is distributive.

Proof. As both $\mathbb{L}D \circ V$ and $V' \circ \mathbb{L}D'$ preserve geometric realizations, it is sufficient to verify that the map $\mathbb{L}D \circ V \to V' \circ \mathbb{L}D'$ is an equivalence when restricted to $F\mathcal{P}$. Here, it follows from smoothness and Proposition 3.3. $\square$

Thus $\mathbb{L}D$ is unambiguous provided F is smooth, for in this case the derived abelianization of an F-model is the same as computed with respect to F or $\mathcal{P}$. In practice we will assume that our algebras are smooth when we consider the Quillen cohomology of their algebras.

We end by noting the following, illustrating the purpose of smooth algebras.

Proposition 3.5. Fix a smooth algebra F. For $B \in \mathcal{M}\mathrm{odel}_F$ and $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{DF}$, there is a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}_{DF}^{q-p}(\mathbb{L}_p D(B), M) \Rightarrow H_F^q(B; M), \qquad d_r^{p,q} \colon E_r^{p,q} \to E_r^{p-r,q+1},$$

which is convergent, for instance, if $\mathbb{L}D(B)$ is truncated. In particular, if B is smooth as an object of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, then

$$H_F^q(B; M) \cong \mathrm{Ext}_{DF}^q(D(B); M).$$

Proof. By definition, $\mathcal{H}_F(B; M) = \mathcal{E}\mathrm{xt}_{DF}(\mathbb{L}D(B), M)$. Smoothness of F ensures that the notation $\mathbb{L}D(B)$ is unambiguous. The spectral sequence is then associated to the filtration of $\mathcal{E}\mathrm{xt}_{DF}(\mathbb{L}D(B), M)$ by the Whitehead tower of $\mathbb{L}D(B)$. $\square$

More explicit examples of these ideas will be given in Subsection 3.3.4.

3.2. Koszul algebras

This section is concerned with additive theories, and in particular with the notion of a *Koszul algebra* over an additive theory. We begin by covering some general topics relevant to homology and cohomology in the additive setting. In particular, in Subsection 3.2.2 we review the cobar complex in detail, in part to be explicit about our conventions. In Subsection 3.2.3, we give the relevant notion of the homology and cohomology of an augmented algebra. In Subsection 3.2.4, we generalize to this setting some standard facts about how homology and cohomology behave with respect to tensor products.

With the above in place, the remaining subsections are dedicated to the Koszul story. In Subsection 3.2.5, we give the definition of a Koszul algebra over an additive theory, and show how this definition immediately implies the existence of Koszul resolutions and Koszul complexes. In Subsection 3.2.6, we begin to make this more explicit by introducing quadratic algebras and using them to describe the cohomology of a homogeneous Koszul algebra. This serves to describe Koszul complexes as graded objects, and in Subsection 3.2.7 we describe their differentials. In Subsection 3.2.8, we explain how the standard PBW criterion for detecting Koszulity translates to our context.

3.2.1. Coalgebras. Fix additive theories $\mathcal{P}$ and $\mathcal{P}'$. To emphasize that we are working in the additive setting, we will refer to $\mathcal{P}'$-$\mathcal{P}$-bimodels as $\mathcal{P}'$-$\mathcal{P}$-*bimodules*; see Example 3.4. In addition, we can extend a $\mathcal{P}'$-$\mathcal{P}$-bimodule $H \colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\mathrm{cn}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}'}^{\mathrm{cn}}$ to a colimit-preserving functor $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}'}$, and will do so without change of notation.

It happens on occasion that a bimodule H has the property that its right adjoint $H^\vee$ preserves colimits; in this case, $H^\vee$ is also a bimodule, with further right adjoint $H^{\vee\vee}$.

Example 3.10. Let A and B be ordinary associative algebras and H a discrete B-A-bimodule. As mentioned in Example 3.4, we can recover the ordinary dual bimodule $\mathrm{L}\mathcal{M}\mathrm{od}_B(H, B)$ of H by considering the restriction of $H^\vee$ to the category of finitely generated free B-modules; its left Kan extension to the category of all free B-modules is then the functor $H_0^\vee(M) = \mathrm{Hom}_B(H, B) \otimes_B M$. There is a comparison map

$$H_0^\vee \to H^\vee, \quad \theta\colon \mathrm{Hom}_B(H, B) \otimes_B M \to \mathrm{Hom}_B(H, M), \quad \theta(f \otimes m)(h) = f(h)m,$$

which is an isomorphism when H is finitely presented and projective as a left B-module. $\triangleleft$

It is not necessary for $H^\vee$ to preserve colimits to talk about monad structures on $H^\vee$. These are equivalently comonad structures on H, and thus deserve to be called $\mathcal{P}$-*coalgebras*. In this case, $H^\vee$-modules are the analogues of H-*contramodules* in the sense of [**EM65**, Section III.5], but we will not use this name.

3.2.2. Cobar complexes. In this subsection we review bar resolutions and cobar complexes in some detail, in part to make our conventions explicit. Fix for the moment an arbitrary category $\mathcal{M}$—not necessarily a 1-category—and a monad T on $\mathcal{M}$. For $M \in \mathcal{A}\mathrm{lg}_T$, we may form the *bar construction* $B(T, T, M)$. This is the simplicial object augmented over M with

$$B_n(T, T, M) = T^{1+n}M, \qquad d_i = T^i m T^{n-i}\colon T^{1+1+n}M \to T^{1+n}M, \quad 0 \leq i \leq n+1.$$

Here, d_{n+1} is to be understood as given by the T-module structure on M.

Lemma 3.7. The simplicial object $B(T, T, M)$ is a resolution of M, in the sense that the augmentation extends to an equivalence

$$\operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} B_n(T, T, M) \simeq M$$

in $\mathcal{A}\mathrm{lg}_T$.

Proof. The augmented simplicial object $M \leftarrow B(T, T, M)$ is T-split, so the claim follows as $\mathcal{A}\mathrm{lg}_T \to \mathcal{M}$ creates T-split geometric realizations [**Lur17a**, Theorem 4.7.3.5]. $\square$

We now restrict ourselves to the case where $\mathcal{M} = \mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ for a discrete additive theory $\mathcal{P}$ and $T = F$ is a discrete $\mathcal{P}$-algebra; the rest of this subsection takes place in a 1-category. Assuming that F is a colimit-preserving monad on a category of the form $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$ is much more than is necessary, as most of the following is just an explicit comparison of finite formulas; the assumption is made purely for notational convenience.

In this additive setting, we may produce from $B(F,F,M)$ the *unreduced bar resolution* $C^{\mathrm{un}}(F,F,M)$, which is a chain complex of F-modules of the form

$$C^{\mathrm{un}}_n(F,F,M) = F^{1+n}M, \qquad d = \sum_{0\le i\le n+1} (-1)^i d_i \colon F^{1+1+n}M \to F^{1+n}M,$$

as well as the *reduced bar resolution* $C(F,F,M)$, which is the quotient chain complex of $C^{\mathrm{un}}(F,F,M)$ with

$$C_n(F,F,M) = FF_+^nM, \qquad F_+ = \mathrm{Coker}(I \to F).$$

Given $M, M' \in \mathrm{L}\mathcal{M}\mathrm{od}_F^{\heartsuit}$, define

$$B_F(M,M') = \mathrm{Hom}_{F\mathcal{P}}(B(F,F,M),M'),$$

so that

$$B_F^n(M,M') = \mathrm{Hom}_{F\mathcal{P}}(F^{1+n}M,M') \cong \mathrm{Hom}_{\mathcal{P}}(F^nM,M').$$

This is a cosimplicial abelian group modeling $\mathcal{E}\mathrm{xt}_F(M,M')$ provided that $B(F,F,M)$ consists of projective F-modules. From this we extract the *unreduced cobar complex* $C_F^{\mathrm{un}}(M,M')$ and *reduced cobar complex* $C_F(M,M') \subset C_F^{\mathrm{un}}(M,M')$; the differential on $C_F^{\mathrm{un}}(M,M')$ is given

$$\begin{aligned}
\delta\colon\ &\mathrm{Hom}_{\mathcal{P}}(F^nM,M') \to \mathrm{Hom}_{\mathcal{P}}(F^{1+n}M,M'),\\
\delta &= \delta_0 + \sum_{1\le i\le n} (-1)^i\delta_i + (-1)^{n+1}\delta_{n+1},\\
\delta_0(f) &= m \circ Ff \colon F^{1+n}M \to FM' \to M',\\
\delta_i(f) &= f \circ F^{i-1}mF^{n-i} \colon F^{1+n}M \to F^nM \to M',\\
\delta_{n+1}(f) &= f \circ F^n m \colon F^{1+n}M \to F^nM \to M'.
\end{aligned}$$

Lemma 3.8. Fix $M, M', M'' \in \mathrm{L}\mathcal{M}\mathrm{od}_F^{\heartsuit}$. Define

$$\wr\colon C_F^{\mathrm{un},n}(M,M') \otimes C_F^{\mathrm{un},n'}(M',M'') \to C_F^{\mathrm{un},n'+n}(M,M'')$$

as follows: given $f\colon F^nM \to M'$ and $f'\colon F^{n'}M' \to M''$, set

$$(-1)^{nn'} f \wr f' = f' \circ F^{n'}f \colon F^{n'+n}M \to F^{n'}M' \to M''.$$

Then $\wr$ has the following properties:

(1) If $f \in C_F^n(M,M')$ and $f' \in C_F^{n'}(M',M'')$, then $f \wr f' \in C_F^{n'+n}(M,M'')$.
(2) $\delta(f \wr f') = \delta(f) \wr f' + (-1)^n f \wr \delta(f')$, and thus $\wr$ passes to pairings

$$\begin{aligned}
C_F^{\mathrm{un}}(M,M') \otimes C_F^{\mathrm{un}}(M',M'') &\to C_F^{\mathrm{un}}(M,M''),\\
C_F(M,M') \otimes C_F(M',M'') &\to C_F(M,M'')
\end{aligned}$$

of cochain complexes.

(3) Suppose that $C(F, F, M)$ and $C(F, F, M')$ are projective resolutions of M and M'. Then the induced pairing

$$\wr\colon \operatorname{Ext}^n_F(M, M') \otimes \operatorname{Ext}^{n'}_F(M', M'') \to \operatorname{Ext}^{n'+n}_F(M, M'')$$

is the graded opposite of the standard Yoneda composition:

$$f \wr f' = (-1)^{nn'} f' \circ f.$$

Here, to be explicit, we take the Yoneda composition as defined in [**Mac67**, Section III.5, Theorem III.6.4] as the standard.

Proof. (1) This is clear.

(2) Fix $f\colon F^n M \to M'$ and $f'\colon F^{n'} M' \to M''$. The main point is that

$$\delta_{n+1}(f') \circ F^{n'+1} f = f' \circ F^{n'} \delta_0(f);$$

this allows us to compute

$$\begin{aligned}
\delta(f' \circ F^{n'} f) &= \sum_{i=0}^{n'+1+n} (-1)^i \delta_i(f' \circ F^{n'} f) \\
&= \sum_{i=0}^{n'} \delta_i(f' \circ F^{n'} f) + (-1)^{n'} \sum_{i=n'+1}^{n'+1+n} (-1)^i \delta_i(f' \circ F^{n'} f) \\
&= \sum_{i=0}^{n'} \delta_i(f') \circ F^{n'} f + (-1)^{n'} \sum_{i=1}^{1+n} (-1)^i f' \circ F^{n'} \delta_i(f) \\
&= \sum_{i=0}^{n'+1} \delta_i(f') \circ F^{n'} f + (-1)^{n'} \sum_{i=0}^{1+n} (-1)^i f' \circ F^{n'} \delta_i(f) \\
&= \delta(f') \circ F^{n'} f + (-1)^{n'} f' \circ F^{n'} \delta(f),
\end{aligned}$$

which yields

$$\begin{aligned}
\delta(f \wr f') &= (-1)^{nn'} \delta(f' \circ F^{n'} f) \\
&= (-1)^{nn'} \delta(f') \circ F^n f + (-1)^{nn'+n'} f' \circ F^{n'} \delta(f) \\
&= (-1)^{nn'+n(n'+1)} f \wr \delta(f') + (-1)^{nn'+n'+(n+1)n'} \delta(f) \wr f' \\
&= \delta(f) \wr f' + (-1)^n f \wr \delta(f')
\end{aligned}$$

as claimed.

(3) We first introduce a bit of local notation. If $\mathcal{A}$ is an additive category, there are shift functors

$$\operatorname{Ch}(\mathcal{A}) \to \operatorname{Ch}(\mathcal{A}), \qquad C \mapsto C[p]$$

for $p \in \mathbb{Z}$ defined on objects by

$$C[p]_n = C_{n-p}, \qquad d_n^{C[p]} = (-1)^p d_{n-p}^C,$$

and on morphisms by

$$f[p]_n = f_{n-p}.$$

Now fix a map $f\colon F^n M \to M'$ in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$. This lifts to a map

$$f^+\colon C^{\mathrm{un}}(F,F,M) \to C^{\mathrm{un}}(F,F,M')[n]$$

of graded objects by

$$f^+_{<n} = 0, \qquad f^+_{k+n} = (-1)^{nk} F^{1+k} f.$$

This satisfies

$$f^+_{k+n} \circ d - d \circ f^+_{1+k+n} = (-1)^{(n+1)k} F^{1+k} \delta(f),$$

and is thus a chain map whenever f is a cocycle. If $f'\colon F^{n'} M' \to M''$ is another map in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, then $f' \circ f^+_{n'+n} = f \wr f'$. We conclude by observing that if $C(F,F,M)$ and $C(F,F,M')$ are projective resolutions of M and M', and f and f' are cocycles, then $f' \circ f^+_{n'+n}$ is a cocycle representing the standard Yoneda composition of the classes represented by f and f', only twisted by $(-1)^{nn'}$. ☐

When $\mathcal{P}$ is the theory of $\mathbb{Z}$-graded modules over an ordinary $\mathbb{Z}$-graded algebra, it is standard practice to insert additional signs in various places when developing the homological algebra of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, where these additional signs are dependent on the internal degrees of elements involved. The internal degrees of elements of a graded module cannot be defined in a Morita-invariant way, so these signs cannot be incorporated at the present level of generality. In practice one can simply modify the constructions of this section to be compatible with whatever conventions are most convenient for a given category. Here is an example indicating the effect of this in the standard case.

Example 3.11. Let k be a commutative ring and A be an ordinary projective $\mathbb{Z}$-graded augmented k-algebra in which k is central. Let H be the cohomology algebra of A defined with conventions from [**Pri70**]. Write s^q for q-fold shift, so that $(s^q M)_n = M_{n-q}$ for a $\mathbb{Z}$-graded object M. Then H is a bigraded object with

$$H^{p,q} = \mathrm{Ext}^p(k, s^q k).$$

Write $\smile$ for the product on H and $\circ$ for the Yoneda composition on Ext. Then for $x \in H^{p,q}$ and $x' \in H^{p',q'}$ these pairings satisfy

$$x' \smile x = (-1)^{q(q'-p')} s^q x' \circ x.$$

Closer to our conventions is the bigraded opposite algebra H^{op}. This is the algebra with $(H^{\mathrm{op}})^p_q = H^{p,-q}$ and product given for $x \in (H^{\mathrm{op}})^p_q$ and $x' \in (H^{\mathrm{op}})^{p'}_{q'}$ by

$$x \smile^{\mathrm{op}} x' = (-1)^{(q-p)(q'-p')} x' \smile x.$$

If we identify $(H^{\mathrm{op}})^p_q = \mathrm{Ext}^p(s^q k, k)$, then this satisfies

$$x \smile^{\mathrm{op}} x' = (-1)^{pq'} s^{q'} x \wr x'.$$

$\triangleleft$

3.2.3. Cohomology of augmented algebras. Fix an additive theory $\mathcal{P}$ and $\mathcal{P}$-algebra F. Suppose that F is *augmented*; that is, that we have chosen a map $\epsilon \colon F \to I$ of algebras, where I is the initial algebra, given by the identity functor. Restriction along the augmentation gives a functor

$$\epsilon^* \colon \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}} \to \mathrm{L}\mathcal{M}\mathrm{od}_F, \qquad \epsilon^*(M) = \overline{M}.$$

As ϵ^* preserves limits and colimits, it is part of an adjoint triple $\epsilon_! \dashv \epsilon^* \dashv \epsilon_*$, giving thus a (non-discrete) $\mathcal{P}$-coalgebra $\epsilon_!\epsilon^*$ with right adjoint monad $\epsilon_*\epsilon^*$. The functor $\epsilon_!\epsilon^*$ can be identified as arising from a bar construction: for $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$, there is an equivalence

$$\epsilon_!\epsilon^* M \simeq \epsilon_! \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} B_n(F, F, \overline{M}) \simeq \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} B_n(I, F, \overline{M}),$$

where by definition $B(I, F, -) = \epsilon_! B(F, F, -)$. Likewise, $\epsilon_*\epsilon^*$ can be identified as

$$(\epsilon_*\epsilon^* M)(P) = \mathcal{E}\mathrm{xt}_{\mathcal{P}}(P, \epsilon_*\epsilon^* M) = \mathcal{E}\mathrm{xt}_F(\overline{P}, \overline{M}),$$

which may be computed via a cobar construction.

We will make minimal use of these homotopical objects directly, but will make use of their algebraic shadows. Specialize now to the case where $\mathcal{P}$ is discrete and F is projective. Under these assumptions, $C(I, F, -)$ is a chain complex of discrete $\mathcal{P}$-bimodules modeling $\epsilon_!\epsilon^*$. Define

$$H_n(F) = \pi_n \epsilon_!\epsilon^* \colon \mathcal{P} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \qquad H_n(F)_{P,P'} = H_n C(I, F, \overline{P})_{P'};$$
$$H^n(F) = \pi_{-n} \epsilon_*\epsilon^* \colon \mathcal{P} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \qquad H^n(F)_{P,P'} = \mathrm{Ext}^n_F(\overline{P}', \overline{P}).$$

These extend to endofunctors of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, and if each $H_n(F)_P$ is projective, then $H_n(F)$ is a bimodule with $H_n(F)^\vee \cong H^n(F)$. The products of Lemma 3.8 give pairings

$$\wr \colon H^{n'}(F)_{P',P''} \otimes H^n(F)_{P,P'} \to H^{n'+n}(F)_{P,P''},$$

and these make $H^*(F)$ into a graded monad. When each $H_n(F)$ is projective, the identification $H_n(F)^\vee \cong H^n(F)$ implies that $H_*(F)$ can be considered as a graded comonad, although we will not make use of this.

Example 3.12. Let k be an ordinary algebra and A an ordinary augmented projective k-algebra. Then treating A as an algebra for the theory of left k-modules, the above definitions give

$$H_*(A)_{k,k} = \mathrm{Tor}^A_*(k,k), \qquad H^*(A)_{k,k} = \mathrm{Ext}^*_A(k,k).$$

Here $H^*(A)_{k,k}$ is itself an augmented k-algebra, with multiplication given by the graded opposite of Yoneda composition. On the other hand, suppose instead that A is an ordinary $\mathbb{Z}$-graded augmented projective k-algebra. Then still we have

$$H^*(A)_{e_p,e_q} = \mathrm{Ext}^*_A(e_q, e_p),$$

where e_a denotes a copy of k in degree a. In particular, we can extract from this a left k-module

$$H^*(A)_{e_0} = \mathrm{Ext}^*_A(e_*, e_0).$$

Heuristically, this is the ordinary cohomology algebra of A. However, note the following subtlety: to make $H^*(A)_{e_0}$ into an ordinary algebra requires the additional structure of the isomorphisms $\mathrm{Ext}^*_A(e_{a+b}, e_b) \cong \mathrm{Ext}^*_A(e_a, e_0)$. $\triangleleft$

3.2.4. Homology and compositions. Classically, the homology of ordinary augmented algebras is well-behaved with respect to base change, and there are Künneth isomorphisms describing the homology a tensor product under suitable flatness conditions. We will need the analogues of these where tensor products are replaced with compositions in the sense of Subsection 3.1.5.

Lemma 3.9. Suppose given a square

$$\begin{tikzcd} \mathcal{D}' \arrow[r, shift left, "V'"] \arrow[d, shift left, "U'"] & \mathcal{D} \arrow[l, shift left, "F'"] \arrow[d, shift left, "U"] \\ \mathcal{C}' \arrow[r, shift left, "V"] \arrow[u, shift left, "T'"] & \mathcal{C} \arrow[l, shift left, "F"] \arrow[u, shift left, "T"] \end{tikzcd}$$

of adjunctions between 1-categories. Then there is a natural simplicial map

$$T'B(F,F,C') \to B(F',F',T'C')$$

defined for $C' \in \mathcal{C}'$, which is an isomorphism if the square is distributive.

Proof. In degree n, this is the map

$$T'F(VF)^nVC' \to F'(V'F')^nV'T'C'$$

obtained by repeated application of the mate

$$T'FV \simeq F'TV \to F'V'T'.$$

That this is an isomorphism when the square is distributive is straightforward. $\square$

Proposition 3.6. Let $\mathcal{C}$ be a 1-category, let T and F be monads on $\mathcal{C}$ together with a distributive law $c\colon FT \to TF$ allowing us to form the composition monad TF, and let

$$\begin{tikzcd} \mathcal{D}' \arrow[r, shift left, "V'"] \arrow[d, shift right, "U'"'] & \mathcal{D} \arrow[l, shift left, "F'"] \arrow[d, shift right, "U"'] \\ \mathcal{C}' \arrow[u, shift right, "T'"'] \arrow[r, shift left, "V"] & \mathcal{C} \arrow[l, shift left, "F"] \arrow[u, shift right, "T"'] \end{tikzcd}$$

be the associated monadic distributive square. Suppose that F is equipped with an augmentation ϵ such that $T\epsilon \circ c = \epsilon T$. Then ϵ lifts to an augmentation on F', and moreover

$$TB(I, F, \epsilon_F^* C) \cong B(I, F', \epsilon_{F'}^* TC)$$

for $C \in \mathcal{C}$.

Proof. The given assumption on the augmentation of F implies that $T\epsilon\colon TF \to T$ is a map of monads, and this gives rise to the augmentation on F'. The claimed isomorphism of bar constructions follows from Lemma 3.9 and the isomorphisms $T'\epsilon_F^* \simeq \epsilon_{F'}^* T$ and $\epsilon_{F'!}T' \simeq T\epsilon_{F!}$. $\square$

Proposition 3.7. Let $\mathcal{P}$ be a discrete additive theory, and fix discrete projective augmented $\mathcal{P}$-algebras T and F. Suppose that we have chosen a distributive law $c\colon FT \to TF$ such that $\epsilon_T\epsilon_F \circ c = \epsilon_F\epsilon_T$. Then the composite monad $T \circ F$ is augmented, and if each $H_n(T)$ is projective, then $H_n(T \circ F) \cong \bigoplus_{i+j=n} H_i(F) \circ H_j(T)$.

Proof. It is easily verified that $\epsilon_T\epsilon_F$ makes TF into an augmented monad. Now write the monadic distributive square associated to the composite TF as

$$\begin{tikzcd} \mathcal{D}' \arrow[r, shift left, "V'"] \arrow[d, shift right, "U'"'] & \mathcal{D} \arrow[l, shift left, "F'"] \arrow[d, shift right, "U"'] \\ \mathcal{C}' \arrow[u, shift right, "T'"'] \arrow[r, shift left, "V"] & \mathcal{C} \arrow[l, shift left, "F"] \arrow[u, shift right, "T"'] \end{tikzcd}.$$

Then there is a natural map

$$\epsilon_{TF!}\epsilon_{TF}^* \simeq \epsilon_{F!}\epsilon_{T'!}\epsilon_{F'}^*\epsilon_T^* \to \epsilon_{F!}\epsilon_F^*\epsilon_{T!}\epsilon_T^*$$

which we claim is an isomorphism; here, these functors are to be interpreted in the derived sense. It is sufficient to verify that this map induces

$$V\epsilon_{T'!}\epsilon_{F'}^* TP \simeq V\epsilon_F^*\epsilon_{T!}TP$$

for $P \in \mathcal{P}$. The right hand side is simply P, and we compute the left hand side to be

$$V\epsilon_{T'!}\epsilon_{F'*}TP \simeq V\epsilon_{T'!}T'\epsilon_F^* P \simeq V\epsilon_F^* P \simeq P.$$

When each $H_n(T)$ is projective, we can split $\epsilon_{T!}\epsilon_T^* P \simeq \bigoplus_{n\geq 0} \Sigma^n H_n(T)_P$ for $P \in \mathcal{P}$. Thus in this case we have

$$\begin{aligned} H_n(T \circ F)_P &= \pi_n \epsilon_{TF!}\epsilon_{TF}^* P \cong \pi_n \epsilon_{F!}\epsilon_F^* \epsilon_{T!}\epsilon_T^* P \\ &\cong \pi_n \bigoplus_{k\geq 0} \Sigma^k \epsilon_{F!}\epsilon_F^* H_k(T)_P \cong \bigoplus_{i+j=n} (H_i(F) \circ H_j(T))_P \end{aligned}$$

as claimed. □

3.2.5. Koszul resolutions. Our goal for the rest of this section is to generalize Priddy's theory of Koszul algebras and Koszul resolutions [**Pri70**] to the setting of algebras over additive theories. The approach we take is strongly influenced by the approach taken in [**Rez12**, Section 4]. For us, the purpose of this theory is to give concrete tools for certain homological computations, and so everything that follows should be interpreted as taking place within a 1-category; in particular all of our theories and algebras are discrete.

Fix an additive theory $\mathcal{P}$ and $\mathcal{P}$-algebra F.

Definition 3.6.

(1) F is a *filtered algebra* if we have chosen a filtration $F = \operatorname{colim}_{n\to\infty} F_{\leq n}$ of F by subbimodules such that
   (a) $I = F_{\leq 0} \subset F$ is the unit;
   (b) The product on F restricts to maps $F_{\leq n} \circ F_{\leq m} \to F_{\leq n+m}$.
(2) F is a *graded algebra* if we have chosen a decomposition $F = \oplus_{n\geq 0} F[n]$ of bimodules such that
   (a) $I = F[0] \subset F[n]$ is the unit;
   (b) The product on F restricts to maps $F[n] \circ F[m] \to F[n+m]$.

   In particular, if F is graded then F is augmented.
(3) The *associated graded algebra* of a filtered algebra F is the graded algebra $\operatorname{gr} F$ given by

$$\operatorname{gr} F = \bigoplus_{m\geq 0} F[m], \qquad F[m] = \operatorname{Coker}(F_{\leq m-1} \to F_{\leq m}),$$

   with multiplication induced by that on F.
(4) F is a *projective filtered algebra* if both F and $\operatorname{gr} F$ are projective. ◁

Suppose now that F is a projective filtered algebra. For $M \in \mathrm{LMod}_{\mathcal{P}}^{\heartsuit}$, there is a filtration $C^{\mathrm{un}}(F, F, M) = \operatorname{colim}_{m\to\infty} C^{\mathrm{un}}(F, F, M)[\leq m]$ obtained by declaring

$$C_n^{\mathrm{un}}(F, F, M)[\leq m] = \operatorname{Im}\left(\bigoplus_{m_1+\cdots+m_n=m} F F_{\leq m_1} \cdots F_{\leq m_n} \to C_n^{\mathrm{un}}(F, F, M) \right),$$

and this induces a filtration on $C(F,F,M)$. In particular, we obtain the associated graded complex

$$\operatorname{gr} C(F,F,M) = \bigoplus_{m\geq 0} C(F,F,M)[m],$$

where by definition $C(F,F,M)[m]$ fits into a short exact sequence

$$0 \to C(F,F,M)[\leq m-1] \to C(F,F,M)[\leq m] \to C(F,F,M)[m] \to 0.$$

LEMMA 3.10. Fix notation as above. Then

(1) $\operatorname{gr} C(F,F,M) = FC(I, \operatorname{gr} F, \overline{M})$;

(2) Explicitly,

$$C_n(F,F,M)[m] = \bigoplus_{\substack{m_1+\cdots+m_n=m\\ m_1,\ldots,m_n\geq 1}} FF[m_1]\cdots F[m_n]M;$$

(3) In particular,

  (a) $C_n(F,F,M)[\leq m] = 0$ for $n > m$;

  (b) $C_n(F,F,M)[\leq n] = C_n(F,F,M)[n] = F[1]^{\circ n}(M)$.

PROOF. Immediate from the definitions. $\square$

If F is augmented, then the above filtration on $C(F,F,M)$ induces a filtration on $C(I,F,M)$. When F is graded, this filtration is split on $C(I,F,\overline{P})$, yielding gradings $H_*(F)_P = H_* \bigoplus_{n\geq 0} C(I,F,\overline{P})[n]$ and $H^*(F) = \prod_{n\geq 0} H^*(F)[n]$.

DEFINITION 3.7. Fix a $\mathcal{P}$-algebra F. Say that F is a *homogeneous Koszul $\mathcal{P}$-algebra* if

(1) F is projective and has been equipped with a grading;

(2) $H_n(F)[m] = 0$ for $n \neq m$.

Say that F is a *Koszul $\mathcal{P}$-algebra* if

(1′) F has been equipped with a projective filtration;

(2′) $\operatorname{gr} F$ is a homogeneous Koszul $\mathcal{P}$-algebra. $\triangleleft$

Now suppose that F is a projective filtered algebra, and fix a $\mathcal{P}$-projective F-module M. The filtration $C(F,F,M) \simeq \operatorname{colim}_{m\to\infty} C(F,F,M)[\leq m]$ gives rise to a spectral sequence of signature

$$E^1_{p,q} = FH_q(\operatorname{gr} F)[p](M) \Rightarrow H_q C(F,F,M), \qquad d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q-1}.$$

LEMMA 3.11. Suppose either of the following is satisfied:

(1) Filtered colimits in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ are exact;

(2) The connectivity of $C(F,F,M)[m]$ goes to ∞ as m goes to ∞.

Then the above spectral sequence converges. In particular, this holds if F is Koszul.

Proof. Abbreviate $C = C(F, F, M)$. As each of C, $C[\leq m]$, and $C[m]$ are bounded below complexes of projectives, the construction of this spectral sequence is compatible with that in Subsection 2.6.1, so we may apply the convergence criteria of Proposition 2.25. The conditions are clearly satisfied in case (1), so consider case (2). Write $C[p, q] = \mathrm{Coker}(C[\leq p-1] \to C[\leq q])$. Condition (a) is automatic, and to verify (b) and (c′) it is sufficient to verify that for any fixed n and p, the map $H_n C[p, p+r] \to H_n C[p, p+r+1]$ is an isomorphism for all sufficiently large r. As $\mathrm{Coker}(C[p, p+r] \to C[p, p+r+1]) = C[p+r+1]$, this follows from the given connectivity assumption. □

Define the chain complex $K(F, F, M)$ by $K_p(F, F, M) = E^1_{p,p}$ as above, with differential obtained from the d^1 differential of this spectral sequence. Then $K(F, F, M)$ is a chain complex of the form

$$FH_0(\mathrm{gr}\, F)(M) \leftarrow FH_1(\mathrm{gr}\, F)(M) \leftarrow FH_2(\mathrm{gr}\, F)(M) \leftarrow \cdots ,$$

or more memorably,

$$K(F, F, M) = FH_*(\mathrm{gr}\, F)(M),$$

and this sits as a subcomplex of the bar resolution of M.

Theorem 3.1. *Let F be a Koszul $\mathcal{P}$-algebra, and let M be an F-module which is projective over $\mathcal{P}$. Then there is a splitting $C(F, F, M) \cong K(F, F, M) \oplus C'$, where C' is a contractible chain complex. In particular,*

$$M \leftarrow K(F, F, M)$$

is a projective resolution of M.

Proof. As F is Koszul, the spectral sequence $FH_q(\mathrm{gr}\, F)[p](M) \Rightarrow H_q C(F, F, M)$ collapses into a projective resolution $K(F, F, M) \to M$. The inclusion $K(F, F, M) \to C(F, F, M)$ is a quasiisomorphism of bounded below complexes of projectives, allowing for the indicated splitting. □

Fix a filtered $\mathcal{P}$-algebra F and F-modules M and M' with M projective over $\mathcal{P}$. Then there is a *Koszul complex*

$$K_F(M, M') = \mathrm{L}\mathcal{M}\mathrm{od}_F(K(F, F, M), M').$$

This is a quotient of the cobar complex $C_F(M, M')$, and models $\mathcal{E}\mathrm{xt}_F(M, M')$ when F is Koszul. We will describe these complexes more explicitly in Subsection 3.2.7.

Example 3.13.

(1) The motivating example of a Koszul algebra is the Steenrod algebra [**Pri70**]. For simplicity, take $\mathcal{A}$ to be the mod 2 Steenrod algebra; then $\mathcal{A}$ is Koszul with respect to

the length filtration on $\mathcal{A}$. The Koszul complex $K_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ can be enriched to a complex of graded vector spaces, and is known as the *lambda algebra*. See Example 3.15 for more on this example.

(2) More generally, there are Steenrod algebras for other forms of mod p cohomology, such as in unstable, motivic, equivariant, synthetic, or other flavors of homotopy theories, and examples suggest that one can expect these to be Koszul as well. These examples require a fairly general notion of Koszul algebra: they are not generally augmented, their coefficients rings do not generally live in their center, and they need not be ordinary graded algebras at all, such as in the equivariant setting where one has additional Mackey functor structure, or the unstable setting where one must account for instability conditions. We will give the unstable example more explicitly in Example 3.16. $\triangleleft$

We end by noting the following stability properties of Koszulity.

LEMMA 3.12. Fix a monadic distributive square

$$\begin{array}{ccc} \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{F'} & \underset{F'}{\overset{V'}{\rightleftarrows}} & \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{\mathcal{P}'} \\ {\scriptstyle T'}\uparrow\downarrow{\scriptstyle U'} & & {\scriptstyle T}\uparrow\downarrow{\scriptstyle U} \\ \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{F} & \underset{F}{\overset{V}{\rightleftarrows}} & \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{\mathcal{P}} \end{array},$$

where $\mathcal{P}$ and $\mathcal{P}'$ are additive theories and F and F' are projective algebras over them. In particular, there is a distributive law $c\colon FT \to TF$, and $\mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{F'} \simeq \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{T\circ F}$ where $T \circ F$ is composition of T and F as a monad on $\mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_{\mathcal{P}}$.

(1) Suppose that F is a projective filtered algebra, with filtration compatible with the distributive law. If F is Koszul over $\mathcal{P}$, then F' is Koszul over $\mathcal{P}'$.
(2) Suppose that F and T are projective filtered algebras, with filtrations compatible with the distributive law. Filter $T \circ F$ by $(T\circ F)_{\leq n} = \mathrm{Im}(\bigoplus_{i+j=n} T_{\leq i} \circ F_{\leq j} \to T \circ F)$, so that $\mathrm{gr}(T\circ F) \cong \mathrm{gr}\, T \circ \mathrm{gr}\, F$. Then $T \circ F$ is Koszul.

PROOF. Given Theorem 3.1, these follow from Proposition 3.6 and Proposition 3.7. $\square$

3.2.6. Quadratic algebras. Our goal in this subsection is to describe the structure and cohomology of homogeneous Koszul algebras. Fix an additive theory $\mathcal{P}$, and let H be a $\mathcal{P}$-bimodule. We can then form the free algebra on H as a type of tensor algebra:

$$TH = \bigoplus_{n\geq 0} T_n H, \qquad T_n H = H^{\circ n},$$

with standard multiplication. The right adjoint to this is

$$\widehat{T}H^\vee = (TH)^\vee = \prod_{n\geq 0} (H^{\circ n})^\vee = \prod_{n\geq 0} H^{\vee\circ n};$$

this also carries an obvious multiplication, but we want the slightly less obvious multiplication, obtained by twisting the identifications

$$\widehat{T}_i(H^\vee) \circ \widehat{T}_j(H^\vee) = H^{\vee\circ i} \circ H^{\vee\circ j} \cong H^{\vee\circ(i+j)} \cong \widehat{T}_{i+j}(H^\vee)$$

by $(-1)^{ij}$; write this multiplication as $\wr$. In fact, these two choices are isomorphic by $x \mapsto (-1)^{\frac{|x|(|x|-1)}{2}} x$, so the distinction is essentially invisible in the examples we will consider; the purpose of this choice is to ensure compatibility with the cobar complex in Theorem 3.2 below.

Given a subfunctor $R \subset H \circ H$, we may form the *quadratic algebra*

$$T(H,R) = T(H)/R = \bigoplus_{n\geq 0} T_n(H,R),$$

$$T_n(H,R) = \operatorname{Coker}\left(\sum_{i+j=n} H^{\circ i-1} \circ R \circ H^{\circ j-1} \to H^{\circ n}\right),$$

with multiplication inherited from TH. Likewise, given some subfunctor $R' \subset H^\vee$, we may form the monad

$$\widehat{T}(H^\vee, R') = \prod_{n\geq 0} \widehat{T}_n(H^\vee, R'),$$

$$\widehat{T}_n(H^\vee, R') = \operatorname{Coker}\left(\sum_{i+j=n} H^{\vee\circ i-1} \circ R' \circ H^{\vee\circ j-1} \to H^{\vee\circ n}\right),$$

with multiplication inherited from $\widehat{T}H^\vee$. This is no longer guaranteed to preserve limits in general, but it will in the cases of relevance to us.

LEMMA 3.13. The following are equivalent:

(1) $T(H,R)$ is projective;
(2) H is projective, and for all $P \in \mathcal{P}$, the map $R_P \to (H\circ H)_P$ admits a splitting.

PROOF. (1)⇒(2). If $T(H,R)$ is projective, then H is projective as $H = T_1(H,R)$ is a summand of $T(H,R)$. The sequence

$$0 \to R \to H \circ H \to T_2(H,R) \to 0$$

is exact, so that as $T_2(H,R)$ is projective, it is levelwise split as claimed.

(2)⇒(1) If H is projective and $R \subset H\circ H$ is levelwise split, then each $H^{\circ i-1}\circ R \circ H^{\circ j-1} \subset H^{\circ n}$ is levelwise split. Thus $T_n(H,R)$ is projective, from which it follows that $T(H,R)$ is projective. □

Call a pair (H, R) satisfying the conditions of Lemma 3.13 a *quadratic datum*. Fix now a quadratic datum (H, R). By projectivity, the short exact sequence

$$0 \to R \to H \circ H \to T_2(H, R) \to 0$$

dualizes to a short exact sequence

$$0 \leftarrow R^\vee \leftarrow H^\vee \circ H^\vee \leftarrow R^\perp \leftarrow 0;$$

the pair $(H^\vee, R^\perp)$ could be called the *dual quadratic datum* to (H, R), though with this name dual quadratic data are not themselves quadratic data.

LEMMA 3.14. Fix a quadratic datum (H, R). Then the monad $\widehat{T}(H^\vee, R^\perp)$ preserves limits, and is thus the right adjoint of a coalgebra.

PROOF. As in the proof of Lemma 3.13, the hypotheses imply that $\widehat{T}(H^\vee, R^\perp)$ is levelwise a summand of $\widehat{T}(H^\vee)$, and this proves the lemma. $\square$

REMARK 3.4. The coalgebra which is left adjoint to $\widehat{T}(H^\vee, R^\perp)$ may be identified explicitly as the coquadratic coalgebra

$$\bigoplus_{n\geq 0}\left(\bigcap_{i+j=n} H^{\circ i-1}\circ R\circ H^{\circ j-1}\right)\subset T(H),$$

but we will not have a chance to make use of this. $\triangleleft$

THEOREM 3.2.

(1) Let (H, R) be a quadratic datum. Then $H^1(T(H, R))[1] \cong H^\vee$, and the inclusion $H^\vee \subset H^*(T(H, R))$ extends to an isomorphism $\widehat{T}(H^\vee, R^\perp) \cong \prod_{n\geq 0} H^n(T(H, R))[n]$ of monads.

(2) Let $F = \bigoplus_{n\geq 0} F[n]$ be a homogeneous Koszul algebra, and let $R = \operatorname{Ker}(F[1] \circ F[1] \to F[2])$. Then $F \cong T(F[1], R)$ and $H^*(F) \cong \widehat{T}(F[1]^\vee, R^\perp)$.

PROOF. (1) Abbreviate $C = C(I, T(H, R), -)$. By Lemma 3.10, there is an isomorphism

$$H_nC[n]\cong \operatorname{Ker}\left(H^{\circ n}\to\bigoplus_{i+j=n} H^{\circ i-1}\circ T_2(M,R)\circ H^{\circ j-1}\right).$$

This is left adjoint to $\widehat{T}_n(H^\vee, R^\perp)$, proving (1) additively, and multiplicative compatibility follow by comparing our choice of product on $\widehat{T}(H^\vee, R^\perp)$ with the construction given in Lemma 3.8.

(2) We must show only $F \simeq T(F[1], R)$, for the remaining claims follow from Koszulity and (1). By construction, the inclusion $F[1] \to F$ extends to a map $T(F[1], R) \to F$ of

algebras, and we must verify that this is an isomorphism. By Koszulity, the sequences

$$\bigoplus_{\substack{i+j=n \\ i,j>0}} F[i] \circ F[j] \to F[n] \to 0$$

$$\bigoplus_{\substack{i+j+k=m \\ i,j,k>0}} F[i] \circ F[j] \circ F[k] \to \bigoplus_{\substack{r+s=m \\ r,s>0}} F[r] \circ F[s] \to F[m]$$

are exact for $n > 1$ and $m > 2$. The first implies that each $F[1]^{\circ n} \to F[n]$ is surjective, and thus so too is $T(F[1], R) \to F$. The second implies that any relation seen in the multiplication $F[r] \circ F[s] \to F[m]$ with $r + s = m$ and either $r > 1$ or $s > 1$ is already generated in relations among $F[i]$ with $i < r$ or $i < s$. Thus $T_n(F[1], R) \to F[n]$ is an injection, so an isomorphism. As $T(F[1], R) \to F$ is a direct sum of isomorphisms, it is itself an isomorphism. $\square$

EXAMPLE 3.14. Let $R = W[[a]]$ where $W = W(\kappa)$ is the ring of 2-typical Witt vectors on a perfect field κ of characteristic 2. Define an R-bimodule $\Gamma[1]$ as follows. As a left R-module, $\Gamma[1] \simeq R\{Q_0, Q_1, Q_2\}$. The right R-module structure is determined by

$$\begin{aligned} Q_i \lambda &= \lambda^\sigma Q_i, \quad \lambda \in W \\ Q_0 a &= a^2 Q_0 - 2aQ_1 + 6Q_2 \\ Q_1 a &= 3Q_0 + aQ_2 \\ Q_2 a &= -aQ_0 + 3Q_1, \end{aligned}$$

where $(-)^\sigma$ is the Frobenius automorphism of W. Let $R \subset \Gamma[1] \circ \Gamma[1]$ by spanned by

$$\begin{aligned} Q_1 Q_0 &= 2Q_2 Q_1 - 2Q_0 Q_2, \\ Q_2 Q_0 &= Q_0 Q_1 + aQ_0 Q_2 - 2Q_1 Q_2. \end{aligned}$$

Now set $\Gamma = T(\Gamma[1], R)$. This is the algebra of additive power operations for a certain Morava E-theory at height $h = 2$ and $p = 2$ computed by Rezk [**Rez08**] (cf. Example 4.10), and is Koszul. By Theorem 3.2, there is an isomorphism $H^*(\Gamma) = \widehat{T}(\Gamma[1]^\vee, R^\perp)$; we can compute this explicitly as follows. As $\Gamma[1]$ is finitely generated and free as a left R-module, $\Gamma[1]^\vee$ is a bimodule, and is finitely generated and free as a right R-module on a basis dual to that of $\Gamma[1]$; write this as $\Gamma[1]^\vee = \{Q^0, Q^1, Q^2\}R$. The left R-module structure is given by

$$\begin{aligned} \lambda Q^i &= Q^i \lambda^\sigma, \quad \lambda \in W \\ aQ^0 &= Q^0 a^2 + 3Q^1 - Q^2 a \\ aQ^1 &= -2Q^0 a + 3Q^2 \\ aQ^2 &= 6Q^0 + Q^1 a. \end{aligned}$$

The subspace $R^\perp \subset \Gamma[1]^\vee \circ \Gamma[1]^\vee$ is spanned by

$$\begin{gathered} Q^0Q^0, \qquad Q^1Q^1, \qquad Q^2Q^2, \qquad Q^1Q^0 + Q^0Q^2, \\ Q^1Q^2 + 2Q^0Q^1, \qquad Q^2Q^1 - 2Q^0Q^2, \\ Q^2Q^0 - 2Q^0Q^1 + Q^0Q^2a. \end{gathered}$$

Thus

$$H^*(\Gamma) \cong \{1, Q^0, Q^1, Q^2, Q^0Q^1, Q^0Q^2\}R,$$

with multiplicative structure determined by the preceding relations. $\triangleleft$

3.2.7. Koszul complexes. Our goal in this section is to describe the Koszul complexes computing Ext over a Koszul algebra. We begin with the homogeneous case.

Fix an additive theory $\mathcal{P}$. Fix a quadratic datum (H, R), and write $F = T(H, R)$. Fix $M, N \in \mathrm{L}\mathcal{M}\mathrm{od}_F^\heartsuit$ with M projective over $\mathcal{P}$. Recall from Subsection 3.2.5 the Koszul complex $K_F(M, N)$, which is a quotient of the cobar complex $C_F(M, N)$ satisfying $K_F^n(M, N) = H^n(F)[n](N)(M)$. These groups are described by Theorem 3.2, and it remains only to describe the Koszul differential. In some cases, this may be determined by analyzing the surjective map $C_F(M, N) \to K_F(M, N)$ directly, but we can also proceed as follows.

Observe first that the algebra structure of $H^*(F)$ gives pairings

$$\wr\colon K_F(M, N) \otimes K_F(N, L) \to K_F(M, L)$$

of graded objects. These are compatible with the pairings of Lemma 3.8, and so are pairings of chain complexes. Observe next that as F is generated by $F[1] = H$, the F-module structure on M is determined by a map $H(M) \to M$, i.e. an element of $H^\vee(M)(M) = K_F^1(M, M)$; write Q^M for this element twisted by -1, and define Q^N in the same way.

Theorem 3.3. The differential on $K_F(M, N)$ is given by

$$\delta\colon K_F^n(M, N) \to K_F^{n+1}(M, N), \qquad \delta(f) = Q^M \wr f - (-1)^n f \wr Q^N.$$

Proof. Recall $C_F^n(M, N) = \mathrm{Hom}_{\mathcal{P}}(F_+^{\circ n}M, N)$, where $F \cong I \oplus F_+$, and recall the differential on $C_F^n(M, N)$ from Subsection 3.2.2, of the form

$$\delta = \delta_0 + \sum_{1 \le i \le n} (-1)^i \delta_i + (-1)^{n+1}\delta_{n+1}\colon C_F^n(M, N) \to C_F^{n+1}(M, N).$$

By construction, the inner sum $\sum_{1 \le i \le n}(-1)^i\delta_i$ is killed by the quotient map $C_F(M, N) \to K_F(M, N)$. On the other hand,

$$\begin{aligned} \delta_0(f) &= m \circ Tf = -(-1)^n f \wr Q^N; \\ \delta_{n+1}(f) &= f \circ T^n m = -(-1)^n Q^M \wr f. \end{aligned}$$

Combining these proves the theorem. $\square$

REMARK 3.5. Somewhat more explicitly, $\delta_0(f) = -(-1)^n f \wr Q^N$ is the composite

$$\begin{aligned} K_F^n(M,N) &= \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(M, H^n(F)(N)) \\ &\to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(M, H^n(F)\circ H^1(F)(N)) \\ &\to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(M, H^{n+1}(F)(N)) = K_F^{n+1}(M,N), \end{aligned}$$

and $\delta_{n+1}(f) = -(-1)^n Q^M \wr F$ is the composite

$$\begin{aligned} K_F^n(M,N) &= \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(M, H^n(F)(N)) \\ &\to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(H_1(F)(M), H^n(F)(N)) \\ &\simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(M, H^1(F)\circ H^n(F)(N)) \\ &\to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(M, H^{1+n}(F)(N)) = K_F^{1+n}(M,N). \end{aligned}$$

In either case, the first map which is not an isomorphism encodes the F-module structure on M or N, and the second map which is not an isomorphism is obtained from the multiplication on $H^*(F)$. $\triangleleft$

Now fix a possibly nonhomogeneous Koszul algebra F, and fix $M, N \in \mathrm{L}\mathcal{M}\mathrm{od}_F^{\heartsuit}$ with M projective over $\mathcal{P}$. As before, there is a Koszul complex $K_F(M,N) = H^*(\mathrm{gr}\, F)(N)(M)$, still a quotient of $C_F(M,N)$, and we would like to identify its differential.

Write $qR = \mathrm{Ker}(F_{\leq 1}\circ F_{\leq 1} \to F_{\leq 2})$, so that $(F_{\leq 1}, qR)$ is a quadratic datum. Observe that $\bigoplus_{m\geq 0} F_{\leq m}$ is a graded algebra, and that the inclusion of $F_{\leq 1}$ extends multiplicatively to $T(F_{\leq 1}, qR) \to \bigoplus_{m\geq 0} F_{\leq m}$.

THEOREM 3.4.

(1) The map $T(F_{\leq 1}, qR) \to \bigoplus_m F_{\leq m}$ is an isomorphism of graded algebras;
(2) $T(F_{\leq 1}, qR)$ is a homogeneous Koszul algebra;
(3) The surjection $T(F_{\leq 1}, qR) \to F$ gives rise to a short exact sequence

$$0 \to H^*(\mathrm{gr}\, F) \to H^*(T(F_{\leq 1}, qR)) \to H^{*-1}(\mathrm{gr}\, F) \to 0,$$

which is split when F is augmented.

In particular, $K_F(M,N) \subset K_{T(F_{\leq 1}, qR)}(M,N)$ is a subcomplex with differential on the target described by Theorem 3.2.

PROOF. (1) Define a finite filtration on each $T_n(F_{\leq 1}, qR)$ by

$$T_n(F_{\leq 1}, qR)[\leq m] = \mathrm{Im}\left(\bigoplus_{\substack{\epsilon_1+\cdots+\epsilon_n = m \\ \epsilon_1,\ldots,\epsilon_n \in \{0,1\}}} F_{\leq \epsilon_1}\circ\cdots\circ F_{\leq \epsilon_n} \to T_n(F_{\leq 1}, qR)\right).$$

The map $T_n(F_{\leq 1}, qR) \to F_{\leq n}$ is compatible with filtrations. As $\operatorname{gr} F$ is quadratic, this map induces an isomorphism $\operatorname{gr} T_n(F_{\leq 1}, qR) \cong \bigoplus_{m \leq n} F[m] \cong \operatorname{gr} F_{\leq n}$ of associated graded bimodules, and is therefore itself an isomorphism.

(2–3) The filtration $T(F_{\leq 1}, qR) = \operatorname{colim}_{m\to\infty} T(F_{\leq 1}, qR)[\leq m]$ is multiplicative, though it need not satisfy $T(F_{\leq 1}, qR)[\leq 0] = I$. As $\operatorname{gr} F$ is quadratic, the associated graded $\operatorname{gr} T(F_{\leq 1}, qR) \cong TI \circ \operatorname{gr} F$ is a "polynomial algebra" on $\operatorname{gr} F$. By Lemma 3.12, $\operatorname{gr} T(F_{\leq 1}, qR)$ is a homogeneous Koszul algebra, and $H^n(\operatorname{gr} T(F_{\leq 1}, qR)) \cong H^n(\operatorname{gr} F) \oplus H^{n-1}(\operatorname{gr} F)$. The spectral sequence $H^*(\operatorname{gr} T(F_{\leq 1}, qR)) \Rightarrow H^*(T(F_{\leq 1}, qR))$ collapses into a two-step filtration of $H^*(T(F_{\leq 1}, qR))$ both proving Koszulity and providing the indicated short exact sequences. □

Example 3.15 ([**Bru88**]). Let $\mathcal{A}$ denote the mod 2 Steenrod algebra, so that $\mathcal{A}$ is Koszul with respect to its length filtration. As $\mathcal{A}$ is an ordinary $\mathbb{Z}$-graded algebra, we can upgrade the Koszul complex $K_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong H^*(\operatorname{gr} \mathcal{A})$ to an ordinary differential graded algebra in graded $\mathbb{F}_2$-modules. This is the mod 2 *lambda algebra*.

Explicitly, the lambda algebra may be computed as follows. The algebra $\mathcal{A}$ is generated by $\mathcal{A}_{\leq 1} = \mathbb{F}_2\{\mathrm{Sq}^n : n \geq 0\}$, subject to the quadratic relations

$$\mathrm{Sq}^{2s-r-1}\mathrm{Sq}^s = \sum_i \binom{r-i-1}{i} \mathrm{Sq}^{2s-1-i}\mathrm{Sq}^{s-r+i}$$

for $r \geq 0$, together with the additional nonquadratic relation $\mathrm{Sq}^0 = 1$. Thus $\operatorname{gr} \mathcal{A}$ and $T(\mathcal{A}_{\leq 1}, qR)$ are generated by Sq^n for $n \geq 1$ and $n \geq 0$ respectively, subject to quadratic relations of the same form. By Theorem 3.2, the ordinary cohomology algebras $H^*(\operatorname{gr} \mathcal{A})$ and $H^*(T(\mathcal{A}_{\leq 1}, qR))$ are generated by elements λ_n dual to Sq^{n+1}, subject to the dual relations

$$\lambda_a \lambda_{2a+b+1} = \sum_j \binom{b-j-1}{j} \lambda_{a+b-j}\lambda_{2a+1+j}$$

for $b \geq 0$; the distinction between them is that λ_{-1} is an element of the former but not the latter.

The standard action of $\mathcal{A}$ on $\mathbb{F}_2$ lifts to the action of $T(\mathcal{A}_{\leq 1}, qR)$ on $\mathbb{F}_2$ where Sq^0 acts by the identity. Now $K_{T(\mathcal{A}_{\leq 1}, qR)}(\mathbb{F}_2, \mathbb{F}_2) = H^*(T(\mathcal{A}_{\leq 1}, qR))$, and Theorem 3.3 shows that the Koszul differential is given by the commutator $[\lambda_{-1}, -]$. Thus the lambda algebra $H^*(\operatorname{gr} \mathcal{A}) \subset H^*(T(\mathcal{A}_{\leq 1}, qR))$ is closed under the commutator with λ_{-1}, and we recover this description of its differential. ◁

3.2.8. The PBW criterion. Fix a quadratic algebra $F = T(H, R)$.

Definition 3.8. An *additive decomposition* of F consists of a decomposition

$$H \simeq \bigoplus_{i \in B} H_i$$

of bimodules, together with a subset S of the set B^* of words in the alphabet B such that the map $\bigoplus_{w \in S} F_w \to F$ is an isomorphism, where if $w = (s_1, \ldots, s_n)$ then $F_w = H_{s_1} \circ \cdots \circ H_{s_n}$. This is a *PBW decomposition* if moreover

(1) A word $w = (s_1, \ldots, s_n)$ lives S if and only if each pair (s_i, s_{i+1}) lives in S;
(2) B is equipped with an order, and so B^* with the lexicographic order, such that for all $w', w'' \in S$, either $w'w'' \in S$ or the composite

$$F_{w'} \circ F_{w'} \to F \circ F \to F \to \bigoplus_{w \leq w'w''} F_w$$

is null. ◁

Suppose now that F is equipped with a PBW decomposition, with notation as in the definition. Abbreviate $C = C(I, F, \overline{P})$ for varying P. For $w \in B^*$, define

$$C_k[\leq w] = \bigoplus_{\substack{w_1, \ldots, w_k \in S \\ w_1 \cdots w_k \leq w}} F_{w_1} \circ \cdots \circ F_{w_k} \subset C_k,$$

and similarly define $C_k[<w]$ and $C_k[w]$. The PBW criterion implies that $C_k[\leq w]$ and $C_k[<w]$ are quotient chain complexes of C, and by construction there are short exact sequences

$$0 \to C[w] \to C[\leq w] \to C[<w] \to 0.$$

Proposition 3.8. Suppose that F is equipped with a PBW decomposition, and fix notation for this as above. Suppose that the lexicographic ordering on B^* is well-founded when restricted to subsets of words of a fixed length, and that $H_*C \to H_* \lim_w C[\leq w]$ is an injection. Then F is Koszul.

Proof. The proof is exactly as in [**Pri70**, Theorem 5.3]; we recall the construction in dual form. Under the given hypotheses, it is sufficient to fix a word w of length m and verify that $C[w]$ is acyclic outside degree m. To that end, one constructs $s\colon C[w]_k \to C[w]_{k+1}$ such that $sd + ds$ is the identity on $C[w]_k$ for $k < m$ as follows. Write $w = (r_1, \ldots, r_m)$, and denote decompositions $w = w_1 \cdots w_k$ by $(r_1, \ldots, r_{n_1}; \ldots; r_{n_{k-1}+1}, \ldots, r_m)$. Then s is defined on a summand indexed by a decomposition $w = w_1 \cdots w_k$ as follows. If this decomposition is of the form $(r_1; \ldots; r_{j-1}; r_j, \ldots, r_{j+l}; \ldots)$ with $l \geq 1$ and $r_i r_{i+1} \notin S$ for $i < j$, then s is given by twisting the identification with the summand indexed by $(r_1; \ldots; r_j; r_{j+1}, \ldots, r_{j+l}; \ldots)$ with a sign of $(-1)^j$. On all other summands, $s = 0$. □

The finiteness conditions of Proposition 3.8 are satisfied in settings where one may reasonably call F a locally finite algebra. On the other hand, there are settings where one may reasonably say that F has a PBW basis which fails to respect the $\mathcal{P}$-bimodule structure of F and therefore does not give rise to a PBW decomposition. In such cases, it may nonetheless be possible to deduce Koszulity by filtering the failure away, such as in the proof of Theorem 3.4.

Example 3.16. Let $\mathcal{A}$ denote the mod 2 Steenrod algebra. Following [**Pri70**, Section 7], $\operatorname{gr}\mathcal{A}$ has a PBW basis of admissibles, and this provides a proof of its Koszulity. Now let $\mathcal{U}$ be the monad on the category on graded $\mathbb{F}_2$-vector spaces whose algebras are the unstable $\mathcal{A}$-modules. Then $\mathcal{U}$ is a quotient algebra of $\mathcal{A}$, for our general definition of an algebra, and the admissible basis of $\operatorname{gr}\mathcal{A}$ projects to a PBW decomposition of $\operatorname{gr}\mathcal{U}$. Thus $\mathcal{U}$ is itself a Koszul algebra. This in fact recovers the unstable lambda algebra:

$$K_{\mathcal{U}}(e_n, e_*) \cong \Lambda(n)$$

as chain complexes, up to choices of grading, where e_a denotes a copy of $\mathbb{F}_2$ in degree a. We will cover a variant of this example in greater detail in Subsection 4.1.5. $\triangleleft$

3.3. Plethories

This section is concerned with a generalization of the biring triples of Tall-Wraith [**TW70**], or plethories of [**BW05**]. We give the definition in Subsection 3.3.1. In Subsection 3.3.2, we introduce the notion of a cobialgebroid, and in Subsection 3.3.3 show how the additive operations on rings over a plethory naturally form a cobialgebroid, at least under a minor flatness assumption. In Subsection 3.3.4, we describe what abelianization looks like for rings over a plethory; this serves as an example of the general theory of Subsection 3.1.9. Nothing here is derived; everything that follows takes place in a 1-category, although we expect that various homotopical analogues exist (cf. Remark 3.3).

3.3.1. Exponential monads. Let $\mathcal{P}$ be a symmetric monoidal additive theory. Write $\mathcal{CR}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ for the category of commutative monoids in $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, so that $\mathcal{CR}\mathrm{ing}_{\mathcal{P}}^{\heartsuit} \simeq \mathcal{M}\mathrm{odel}_{S\mathcal{P}}^{\heartsuit}$ where $SP = \bigoplus_{n\geq 0} P^{\otimes n}/\Sigma_n$. We refer to the objects of $\mathcal{CR}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ as $\mathcal{P}$-rings.

Lemma 3.15. The following concepts are equivalent:

(1) Colimit-preserving monads T on $\mathcal{CR}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$, i.e. $S\mathcal{P}$-algebras;
(2) Monads $\mathbb{T}$ on $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ which preserve filtered colimits and reflexive coequalizers and which are equipped with the structure of a strong monoidal functor $\mathbb{T}\colon (\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \oplus) \to (\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \otimes)$ in such a way that for all $X, Y \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, the dashed arrow in

$$\begin{tikzcd}
\mathbb{T}(\mathbb{T}(X) \otimes \mathbb{T}(Y)) \arrow[r, "\simeq"] \arrow[d, dashed] & \mathbb{T}\mathbb{T}(X \oplus Y) \arrow[d] \\
\mathbb{T}(X) \otimes \mathbb{T}(Y) \arrow[r, "\simeq"] & \mathbb{T}(X \oplus Y)
\end{tikzcd}$$

endows $\mathbb{T}(X) \otimes \mathbb{T}(Y)$ with the structure of a $\mathbb{T}$-algebra.

Proof. This is implicit in [**Rez09**]. Given the monad T, one can verify that $\mathbb{T} = T \circ S$ has the indicated structure. Conversely, given $\mathbb{T}$, any $B \in \mathcal{A}\mathrm{lg}_{\mathbb{T}}^{\heartsuit}$ is naturally an object of $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ via the map

$$B \otimes B \to \mathbb{T}(B) \otimes \mathbb{T}(B) \simeq \mathbb{T}(B \oplus B) \to \mathbb{T}(B) \to B,$$

giving a monadic functor $\mathcal{A}\mathrm{lg}_{\mathbb{T}}^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ which preserves sifted colimits. One can then verify that for $A, B \in \mathcal{A}\mathrm{lg}_{\mathbb{T}}^{\heartsuit}$, we have $A \coprod B = A \otimes B$, with $\mathbb{T}$-algebra structure analogous to the diagram in (2), and thus $\mathcal{A}\mathrm{lg}_{\mathbb{T}}^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ preserves all colimits. $\square$

Monads as in Lemma 3.15 are called *exponential monads*.

Definition 3.9. The equivalent data of Lemma 3.15 is a $\mathcal{P}$-*plethory*. $\triangleleft$

We will generally treat a $\mathcal{P}$-plethory as its underlying exponential monad. Given a $\mathcal{P}$-plethory Λ, we will make use of the notation $\Lambda_{P,P'} = \Lambda(P)(P')$ for $P, P' \in \mathcal{P}$. We write $\mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit} = \mathcal{A}\mathrm{lg}_{\Lambda}^{\heartsuit}$, and refer to its objects as Λ-rings.

Among the main pieces of structure of a $\mathcal{P}$-plethory Λ are maps

$$\begin{aligned}
\Delta^{+}&\colon \Lambda_P \to \Lambda_{P \oplus P} \simeq \Lambda_P \otimes \Lambda_P;\\
\epsilon^{+}&\colon \Lambda_P \to \Lambda_0 \simeq \mathbb{1};\\
\Delta^{\times}&\colon \Lambda_{P \otimes P'} \to \Lambda_P \otimes \Lambda_{P'};\\
\epsilon^{\times}&\colon \Lambda_{\mathbb{1}} \to \mathbb{1}.
\end{aligned}$$

Here, Δ^{+} and ϵ^{+} come from the diagonal $P \to P \oplus P$ and unique map $P \to 0$. The map $\Delta^{\times}$ is equivalent to a natural transformation $\mathrm{ev}_P \times \mathrm{ev}_{P'} \to \mathrm{ev}_{P \otimes P'}$, and classifies the multiplication present on Λ-rings, and likewise $\epsilon^{\times}$ classifies the multiplicative identity. In fact these maps are present given just the underlying $S\mathcal{P}$-bimodel of Λ, only one can no longer interpret them as corresponding to natural operations.

3.3.2. Cobialgebroids. Fix a symmetric monoidal additive theory $\mathcal{P}$.

Definition 3.10. A (discrete) $\mathcal{P}$-*cobialgebroid* is a $\mathcal{P}$-algebra Γ together with a lift of Γ to a monoid in the category of oplax symmetric monoidal endofunctors of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, or equivalently, with a lift of $\Gamma^{\vee}$ to a comonoid in the category of lax symmetric monoidal endofunctors of $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$. $\triangleleft$

Denote the category of $\mathcal{P}$-cobialgebroids by $\mathrm{co}\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}_{\mathcal{P}}^{\heartsuit}$.

LEMMA 3.16. Let $\mathcal{C}$ be a symmetric monoidal 1-category, and let $U\colon \mathcal{D} \to \mathcal{C}$ be a plethystic functor with associated monad F and comonad $F^{\vee}$. The following are equivalent:

(1) The structure of a symmetric monoidal category on $\mathcal{D}$ together with the structure of a strong symmetric monoidal functor on U;
(2) A lift of F to a monoid in oplax symmetric monoidal endofunctors of $\mathcal{C}$;
(3) A lift of $F^{\vee}$ to a comonoid in lax symmetric monoidal endofunctors of $\mathcal{C}$.

PROOF. The equivalence of (2) and (3) is clear, so we consider their relation with (1). Given the data of (2), we can make $\mathcal{D} = \mathcal{A}\mathrm{lg}_F$ into a symmetric monoidal category, where for F-algebras A and B, their tensor product is $A \otimes B$ with F-algebra structure $F(A\otimes B) \to F(A) \otimes F(B) \to A \otimes B$. This is seen to refine to the data of (1). Suppose then we have been given the data of (1). As U is strong monoidal, there is for $A, B \in \mathcal{C}$ a map $F(A\otimes B) \to F(A)\otimes F(B)$ adjoint to $A\otimes B \to UF(A)\otimes UF(B) \simeq U(F(A)\otimes F(B))$. This is seen to refine to the data of (2). $\square$

REMARK 3.6. Let Γ be a $\mathcal{P}$-algebra. Then one can be more explicit about the structure necessary to upgrade Γ to a $\mathcal{P}$-cobialgebroid: to lift Γ to an oplax symmetric monoidal functor, we require maps

$$\Gamma(M\otimes N) \to \Gamma(M)\otimes\Gamma(N), \qquad \Gamma(\mathbb{1}) \to \mathbb{1},$$

natural in M and N and subject to the evident counity, coassociativity, and cocommutativity conditions, and for this to make Γ into a $\mathcal{P}$-cobialgebroid we further require that the product $\Gamma\circ\Gamma \to \Gamma$ respects this structure. $\triangleleft$

REMARK 3.7. Let Γ be a $\mathcal{P}$-cobialgebroid, and let A be a monoid in the monoidal category $\mathcal{M}\mathrm{odel}_{\Gamma}^{\heartsuit}$. In particular, A overlies a monoid in $\mathcal{M}\mathrm{odel}_{\mathcal{P}}^{\heartsuit}$, giving $A\otimes -$ the structure of a monad. There is a distributive law of Γ across $A\otimes -$ given by the composite

$$\Gamma(A\otimes M) \to \Gamma(A)\otimes\Gamma(M) \to A\otimes\Gamma(M),$$

and this rise to a composite monad $A\otimes\Gamma$. Algebras for this monad are exactly modules over the monoid A in $\mathcal{M}\mathrm{odel}_{\Gamma}^{\heartsuit}$. This generalizes Example 3.7. $\triangleleft$

We will write $\mathcal{R}\mathrm{ing}_{\Gamma}^{\heartsuit}$ for the category of commutative monoids in $\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma}^{\heartsuit}$. Observe the forgetful functor $\mathcal{R}\mathrm{ing}_{\Gamma}^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ is plethystic.

EXAMPLE 3.17. Let R be an ordinary commutative ring, $\mathcal{R}$ be the theory of R-modules with its usual symmetric monoidal structure, and F be an $\mathcal{R}$-cobialgebroid. Unwinding the definitions, we see this amounts to the following. First, as F is a bimodule, we can write

$F(M) = \Gamma \otimes M$ for an ordinary R-bimodule Γ. Abbreviate $\otimes = \otimes_R$, and use subscripts l and r to denote tensoring with respect to the left or right R-module structure on Γ. Then there are left R-module maps

$$\begin{aligned} m\colon \Gamma\,{}_r\!\otimes_l \Gamma &\to \Gamma; \\ \epsilon^\times\colon \Gamma &\to R; \\ \Delta^\times\colon \Gamma &\to \Gamma\,{}_l\!\otimes_l \Gamma. \end{aligned}$$

The map m makes Γ into an R-algebra, and $\epsilon^\times$ with $\Delta^\times$ satisfy the evident counity, coassociativity, and cocommutativity conditions. The map $\Delta^\times$ is required in addition to be a map of right R-modules with respect to the two right R-module structures on the target given by the action of R on the left and right factor. This corresponds to the fact that $F(M \otimes N) \to F(M) \otimes F(N)$ is natural in both variables, and is what is necessary to extend $\Delta^\times$ to the natural transformation

$$\Gamma\,{}_r\!\otimes M \otimes N \to (\Gamma\,{}_r\!\otimes M)\,{}_l\!\otimes_l(\Gamma\,{}_r\!\otimes N), \qquad \gamma \otimes m \otimes n \mapsto \sum \gamma^\times_{(1)} \otimes m \otimes \gamma^\times_{(2)} \otimes n.$$

Compatibility of m with $\epsilon^\times$ amounts to asking for

$$\epsilon^\times(1) = 1, \qquad \epsilon^\times(\gamma\gamma') = \epsilon^\times(\gamma \cdot \epsilon^\times(\gamma')),$$

and compatibility of m with $\Delta^\times$ amounts to asking for

$$\sum \gamma^\times_{(1)}\gamma'^\times_{(1)} \otimes \gamma^\times_{(2)}\gamma'^\times_{(2)} = \sum (\gamma\gamma')^\times_{(1)} \otimes (\gamma\gamma')^\times_{(2)}$$

in $\Gamma\,{}_l\!\otimes_l \Gamma$. In the above, we have written $\Delta^\times(\gamma) = \sum \gamma^\times_{(1)} \otimes \gamma^\times_{(2)}$. We find that Γ is a *twisted R-bialgebra*, for instance as discussed in [**BW05**, Section 9]. A Γ-ring is a commutative R-ring A equipped with an action of Γ such that $\gamma \cdot (aa') = \sum(\gamma^\times_{(1)} \cdot a)(\gamma^\times_{(2)} \cdot a')$ for all $a, a' \in A$ and $\gamma \in \Gamma$. ◁

Example 3.18. Let Γ be the R-algebra of Example 3.14. Then Γ is an R-cobialgebroid, with augmentation

$$\epsilon^\times(Q_0) = 1, \qquad \epsilon^\times(Q_1) = 0, \qquad \epsilon^\times(Q_2) = 0,$$

and coproduct

$$\begin{aligned} \Delta^\times(Q_0) &= Q_0 \otimes Q_0 + 2Q_1 \otimes Q_2 + 2Q_2 \otimes Q_1; \\ \Delta^\times(Q_1) &= Q_0 \otimes Q_1 + Q_1 \otimes Q_0 + aQ_1 \otimes Q_2 + aQ_2 \otimes Q_1 + 2Q_2 \otimes Q_2; \\ \Delta^\times(Q_2) &= Q_0 \otimes Q_2 + Q_2 \otimes Q_0 + Q_1 \otimes Q_1 + aQ_2 \otimes Q_2. \end{aligned}$$

Thus LMod_Γ is a symmetric monoidal category with underlying tensor product $\otimes_R$, where if M and N are left Γ-modules, then $M \otimes_R N$ has left Γ-module structure

$$\begin{aligned}
Q_0(m\otimes n) &= Q_0(m)\otimes Q_0(n) + 2Q_1(m)\otimes Q_2(n) + 2Q_2(m)\otimes Q_1(n);\\
Q_1(m\otimes n) &= Q_0(m)\otimes Q_1(n) + Q_1(m)\otimes Q_0(n)\\
&\qquad + aQ_1(m)\otimes Q_2(n) + aQ_2(m)\otimes Q_1(n) + 2Q_2(m)\otimes Q_2(n);\\
Q_2(m\otimes n) &= Q_0(m)\otimes Q_2(n) + Q_2(m)\otimes Q_0(n) + Q_1(m)\otimes Q_1(n) + aQ_2(m)\otimes Q_2(n).
\end{aligned}$$

We will continue this example in Example 3.18 and Example 4.7. $\triangleleft$

3.3.3. Additive operations. Fix a symmetric monoidal additive theory $\mathcal{P}$. The primary feature that distinguishes $\mathcal{P}$-plethories from more ordinary algebraic structures such as $\mathcal{P}$-algebras is the presence of nonlinear structure, and in dealing with $\mathcal{P}$-plethories one wants to avoid dealing with this nonlinear structure by any means possible. The use of $\mathcal{P}$-cobialgebroids is one thing that enables this.

There is a purely formal means by which one can extract from any $\mathcal{P}$-plethory a $\mathcal{P}$-cobialgebroid. First, going in the other direction, if Γ is a $\mathcal{P}$-cobialgebroid, then as $\mathcal{R}\mathrm{ing}_\Gamma^\heartsuit \to \mathrm{C}\mathcal{R}\mathrm{ing}_\mathcal{P}^\heartsuit$ is plethystic, it is associated to some $\mathcal{P}$-cobialgebroid $S\Gamma$. This defines a functor $S\colon \mathrm{co}\mathcal{B}\mathrm{i}\mathcal{A}\mathrm{lg}_\mathcal{P}^\heartsuit \to \mathcal{P}\mathrm{leth}_\mathcal{P}^\heartsuit$, which can be described more explicitly as follows: if Γ is a $\mathcal{P}$-cobialgebroid, then as $\Gamma^\vee$ is lax symmetric monoidal, it passes to a limit-preserving comonad $S\Gamma^\vee$ on $\mathcal{R}\mathrm{ing}_\mathcal{P}^\heartsuit$, and this is right adjoint to the colimit-preserving monad $S\Gamma$ on $\mathcal{R}\mathrm{ing}_\mathcal{P}^\heartsuit$. This functor S preserves colimits, and therefore has a right adjoint sending a plethory Λ to a $\mathcal{P}$-cobialgebroid $\Gamma(\Lambda)$. We do not know if $\Gamma(\Lambda)$ admits a nice description in general, but it will under certain extra flatness conditions.

Now fix a $\mathcal{P}$-plethory Λ. As in Proposition 3.1, we may identify

$$\Lambda_{P_1\oplus\cdots\oplus P_n,P} \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit,\mathcal{S}\mathrm{et})}(\mathrm{ev}_{P_1}\times\cdots\times\mathrm{ev}_{P_n},\mathrm{ev}_P).$$

Let $\Gamma_{(P_1,\ldots,P_n),P} \subset \Lambda_{P_1\oplus\cdots\oplus P_n,P}$ be the subset consisting of those operations which are n-multilinear. By allowing $P_1,\ldots,P_n,P$ to vary, this construction can be seen as a functor $\mathcal{P}^{\times n} \to \mathrm{L}\mathcal{M}\mathrm{od}_\mathcal{P}^\heartsuit$.

Lemma 3.17. The $\mathcal{P}$-module $\Gamma_{(P_1,\ldots,P_n)}$ is isomorphic to the total fiber of the n-cube obtained by tensoring together the maps

$$\Delta^+ - \eta\otimes P_i - P_i\otimes\eta\colon \Lambda_{P_i} \to \Lambda_{P_i\oplus P_i} \simeq \Lambda_{P_i}\otimes\Lambda_{P_i}.$$

Proof. For purely notational convenience, we consider the case $n=1$; the general case is identical in nature. Here we are claiming that there is an equalizer diagram

$$\Gamma_P \longrightarrow \Lambda_P \underset{\eta\otimes P + P\otimes\eta}{\overset{\Delta^+}{\rightrightarrows}} \Lambda_{P\oplus P} \ .$$

Evaluating on $P' \in \mathcal{P}$, this is asking for an equalizer diagram

$$\begin{tikzcd} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{M}\mathrm{odel}_\Lambda^\heartsuit, \mathcal{A}\mathrm{b})}(\mathrm{ev}_P, \mathrm{ev}_{P'}) \arrow[d] \\ \operatorname{Hom}_{\operatorname{Fun}(\mathcal{M}\mathrm{odel}_\Lambda^\heartsuit, \mathcal{S}\mathrm{et})}(\mathrm{ev}_P, \mathrm{ev}_{P'}) \arrow[d, shift right, "\eta\otimes P+P\otimes\eta"'] \arrow[d, shift left, "\Delta^+"] \\ \operatorname{Hom}_{\operatorname{Fun}(\mathcal{M}\mathrm{odel}_\Lambda^\heartsuit, \mathcal{S}\mathrm{et})}(\mathrm{ev}_P \times \mathrm{ev}_P, \mathrm{ev}_{P'}) \end{tikzcd} .$$

If we fix a natural operation $\sigma \colon \mathrm{ev}_P \to \mathrm{ev}_{P'}$, then

$$\Delta^+(\sigma)(x,y) = \sigma(x+y), \qquad (\eta\otimes P + P\otimes \eta)(x,y) = \sigma(x)+\sigma(y),$$

and these agree precisely when σ is additive. $\square$

In particular, there are maps $\Gamma_{P_1} \otimes \cdots \otimes \Gamma_{P_n} \to \Gamma_{(P_1,\ldots,P_n)}$. Call Λ *good* if this is an isomorphism for all $P_1,\ldots,P_n \in \mathcal{P}$, and moreover $\Gamma = \Gamma_{(-),(=)}$ is a $\mathcal{P}$-bimodule.

Theorem 3.5. Suppose that Λ is good. Then the $\mathcal{P}$-plethory structure of Λ restricts to a $\mathcal{P}$-cobalgebroid structure on Γ, and $\Gamma(\Lambda) \cong \Gamma$.

Proof. We begin by building the $\mathcal{P}$-cobialgebroid structure on Γ. By hypothesis, Γ is a $\mathcal{P}$-bimodule. As additive operations are closed under composition, we have for $P, P', P'' \in \mathcal{P}$ a bilinear composition map $\Gamma_{P',P''} \times \Gamma_{P,P'} \to \Gamma_{P,P''}$, and these together with the identity maps make Γ into a $\mathcal{P}$-algebra. The counit of Γ is the composite

$$\epsilon^\times \colon \Gamma_{\mathbb{1}} \subset \Lambda_{\mathbb{1}} \to \mathbb{1}.$$

For the coproduct on Γ, consider the diagram

$$\begin{tikzcd} \Gamma_{P\otimes P'} \arrow[r] \arrow[d, dashed, "\Delta^\times"] & \Lambda_{P\otimes P'} \arrow[d, "\Delta^\times"] \\ \Gamma_P \otimes \Gamma_{P'} \arrow[r] & \Lambda_P \otimes \Lambda_{P'} \end{tikzcd} .$$

The right vertical map classifies multiplication for Λ-rings, so the clockwise composite lands in $\Gamma_{(P,P')} \subset \Lambda_{P\oplus P'} \cong \Lambda_P \otimes \Lambda'_P$. This is isomorphic to $\Gamma_P \otimes \Gamma_{P'}$ by assumption, so there is a lift through the dashed map. The axioms of counity, coassociativity, cocommutativity, and compatibility with composition follow from the corresponding facts about operations on Λ-rings. It remains to verify that $\Gamma \cong \Gamma(\Lambda)$. Observe first that the functor S on $\mathcal{P}$-cobialgebroids satisfies, as the notation suggests, $(S\Gamma)(P) = S(\Gamma(P))$. Indeed, there are isomorphisms

$$\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}((S\Gamma)(P), R) \cong (S\Gamma^\vee)(R)(P) \cong \Gamma^\vee(R)(P) \cong \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}(\Gamma(P), R),$$

so that $(S\Gamma)(P)$ has the necessary universal property. It follows from the definition of Γ that the composite $\Gamma(\Lambda) \to S\Gamma(\Lambda) \to \Lambda$ factors uniquely through $\Gamma \subset \Lambda$, so $S\Gamma(\Lambda) \to \Lambda$ factors uniquely through $S\Gamma \to \Lambda$. So $S\Gamma \to \Lambda$ has the necessary universality property to be the counit of the adjunction, giving $S\Gamma \cong S\Gamma(\Lambda)$, from which it follows that $\Gamma \cong \Gamma(\Lambda)$. $\square$

From here on, we will always assume that our plethories are good when we speak of their associated cobialgebroids, as this will be the case for the examples we are interested in. In fact, all of our examples will be *very good*: the inclusion $\Gamma \to \Lambda$ will be levelwise additively split. This ensures that the goodness property of Λ is preserved under various operations, such as base change.

Remark 3.8. Let Λ be a $\mathcal{P}$-plethory and $B \in \mathcal{R}\mathrm{ing}_\Lambda^\heartsuit$. In particular, $B \in \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^\heartsuit$, so there is a category $\mathrm{L}\mathcal{M}\mathrm{od}_B^\heartsuit$ of left B-modules. The forgetful functor $B/\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit \to B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^\heartsuit$ is plethystic, so realizing $B/\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit$ as the category of rings for a B-plethory $B \otimes \Lambda$. The monadic structure is as given by Example 3.8. There is a diagram

$$\begin{tikzcd} \Gamma(B \otimes \Lambda) & B \arrow[l] \\ \Gamma(\Lambda) \arrow[u] & I \arrow[l] \arrow[u] \end{tikzcd}$$

of $\mathcal{P}$-algebras, where B stands for the $\mathcal{P}$-algebra $B \otimes -$, which extends to a map

$$B \otimes \Gamma(\Lambda) \to \Gamma(B \otimes \Lambda)$$

of algebras, which is an isomorphism in the nice cases we will consider. Here, $B \otimes \Gamma(\Lambda)$ has algebra structure as indicated in Remark 3.7. See Example 3.19 for an explicit example of this. $\triangleleft$

3.3.4. Cotangent spaces. Fix a symmetric monoidal additive theory $\mathcal{P}$ and a $\mathcal{P}$-plethory Λ. We are interested in the cohomology of Λ-rings. In particular, we need to identify the relevant categories of abelian group objects. This is as indicated by the general theory of Subsection 3.1.9, but it is worth making this more explicit.

We first review the essentially classical case where $\Lambda = S$ so that $\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit \simeq \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^\heartsuit$. Given $B \in \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^\heartsuit$ and $M \in \mathrm{L}\mathcal{M}\mathrm{od}_B^\heartsuit$, one may form the square-zero extension $B \ltimes M \in \mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^\heartsuit/B)$. As an object of $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^\heartsuit = \mathrm{C}\mathcal{M}\mathrm{on}(\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^\heartsuit)$, this is given by $B \ltimes M =$

$B \oplus M$ with multiplication

$$\begin{aligned}(B \oplus M) \otimes (B \oplus M) &\simeq B \otimes B \oplus B \otimes M \oplus M \otimes B \oplus M \otimes M \\ &\simeq B \otimes B \oplus B \otimes M \oplus B \otimes M \oplus M \otimes M \to B \oplus M\end{aligned}$$

arising from the multiplication on B, the B-module structure of M, and killing $M \otimes M$. This has obvious structure as an object of $B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$, and structure as an abelian group object therein of

$$(B \ltimes M) \times_B (B \ltimes M) \cong B \oplus M \oplus M \to B \oplus M, \qquad ((b, m'), (b, m'')) \mapsto (b, m' + m'').$$

Lemma 3.18. The above construction describes an equivalence between the following categories:

(1) The category $\mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}$;
(2) The full subcategory of $B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$ spanned by the square-zero extensions of B;
(3) The category $\mathcal{A}\mathrm{b}(\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B) \simeq \mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$.

Moreover,

(4) Abelianization $D_B\colon B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit} \to \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}$ is given by $D_B(A) = B \otimes_A \Omega_{A|\mathcal{P}}$, where $\Omega_{A|\mathcal{P}} = J/J^2$, where $J = \mathrm{Ker}(A \otimes A \to A)$;
(5) Abelianization $Q_B\colon B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B \to \mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}$ is given by $Q_B(A) = I/I^2$, where $I = \mathrm{Ker}(A \to B) = \mathrm{Coker}(B \to A)$.

Proof. These statements are standard in the case where $\mathcal{P}$ is the theory of commutative rings, and the same proofs carry over. That $M \mapsto B \ltimes M$ yields an equivalence from $\mathrm{L}\mathcal{M}\mathrm{od}_B^{\heartsuit}$ to the category of square-zero extensions of B is clear, with inverse sending a square-zero extension to its augmentation ideal. The categories $\mathcal{A}\mathrm{b}(\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$ and $\mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$ are equivalent as abelian groups are pointed. The category $\mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$ may be identified as the category of square-zero extensions of B as follows. Fix $A \in \mathcal{A}\mathrm{b}(B/\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B)$. Additively we may split $A = B \oplus M$, and we must show that in fact $A \cong B \ltimes M$ multiplicatively. Because A is a B-ring, it follows that M is a B-module, and the multiplication on A is necessarily determined by the multiplication on B, the B-module structure of M, and some map $M \otimes M \to M$ which we must show is zero. Fixing $P, P' \in \mathcal{P}$, we must show that the map $m\colon M(P) \otimes M(P') \to M(P \otimes P')$ is zero. Write $\mu\colon A \times_B A \to A$ for the abelian group object multiplication on A; we will only use the fact that μ is a unital pairing. As μ is unital, the maps $\mu\colon A(P) \times_{B(P)} A(P) \to A(P)$ satisfy $\mu(x, 0) = x = \mu(0, x)$ for $x \in M(P)$. Thus for $x \in M(P)$ and $x' \in M(P')$, we may compute $m(x \otimes x') = m(\mu(x, 0) \otimes \mu(0, x')) = \mu(m(x, 0) \otimes m(0, x')) = \mu(0, 0) = 0$.

So we have shown the categories of (1)–(3) to be equivalent, and it remains only to verify the claims of (4) and (5). Write D_B and Q_B for the abelianization functors in question, and

D'_B and Q'_B for their proposed descriptions. Then $D_B(A) = Q_B(B \otimes A)$ for $A \in \mathcal{CR}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$, and we claim first that also $D'_B(A) = Q'_B(B \otimes A)$. By definition, $Q_B(B \otimes A) = I/I^2$ where $I = \mathrm{Ker}(B \otimes A \to B) = \mathrm{Ker}(B \otimes_A (A \otimes A) \to B \otimes_A A)$. As $A \otimes A \to A$ admits an A-linear splitting, we can pull B out to get $I = B \otimes_A J$ where $J = \mathrm{Ker}(A \otimes A \to A)$. Thus $D'_B(A) = B \otimes_A (J/J^2) = I/I^2 = Q'_B(B \otimes A)$ as claimed. So it is sufficient to verify just that $Q_B = Q'_B$. Fix $A \in B/\mathcal{CR}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$, and write $A = B \oplus I$ additively, so that $Q'_B(A) = I/I^2$. Let $B \ltimes M$ be some square-zero extension of B. Then maps $A \to B \ltimes M$ in $B/\mathcal{CR}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}/B$ are equivalent to maps $I \to M$ of nonunital B-rings. As M has trivial multiplication, this factors uniquely through the quotient nonunital ring I/I^2, which has trivial multiplication. We find that the quotient ring $B \ltimes I/I^2$ of A is the square-zero extension of B associated to $Q_B(A)$, and thus $Q_B(A) = I/I^2 = Q'_B(A)$. □

We now turn to considering a general $\mathcal{P}$-plethory Λ. Observe that $\mathbb{1}$, the unit of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, is naturally a Λ-ring by the unique map $\mathbb{1} \to \Lambda^{\vee}(\mathbb{1})$ of $\mathcal{P}$-rings. Equivalently, $\mathbb{1}$ is a Λ-ring by the fact that $\mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit} \to \mathcal{CR}\mathrm{ing}_{\mathcal{P}}^{\heartsuit}$ preserves the empty colimit. It follows that there is a good category of $\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit} = \mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit}/\mathbb{1}$ of augmented Λ-rings. For all $P \in \mathcal{P}$, the map $P \to 0$ of $\mathcal{P}$-modules gives a map $\Lambda_P \to \Lambda_0 = \mathbb{1}$ of Λ-rings, so we can regard each Λ_P as an object of $\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit}$. Define now

$$\Delta(\Lambda)\colon \mathcal{P} \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}, \qquad \Delta(\Lambda)_{P,P'} = Q(\Lambda_P)_{P'}.$$

We call $\Delta(\Lambda)$ the *cotangent algebra* of Λ. The functor $\Delta(\Lambda)$ preserves coproducts, so can be regarded as a $\mathcal{P}$-bimodule; it is a $\mathcal{P}$-algebra by the following.

Theorem 3.6.

(1) The category $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit})$ is equivalent to the full subcategory of $\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit}$ spanned by those Λ-rings whose underlying $\mathcal{P}$-ring is a square-zero extension of $\mathbb{1}$. Moreover, the diagram

$$\begin{tikzcd} \mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit}) \arrow[r] \arrow[d] & \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit} \arrow[d] \\ \mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit} \arrow[r] & \mathcal{CR}\mathrm{ing}_{\mathcal{P}}^{\mathrm{aug},\heartsuit} \end{tikzcd}$$

is distributive.

(2) The underlying $\mathcal{P}$-bimodule of the $\mathcal{P}$-algebra associated to the plethystic forgetful functor $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{\Lambda}^{\mathrm{aug},\heartsuit}) \to \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$ is given by $\Delta(\Lambda)$.

(3) Fix $B \in \mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit}$, so that $B/\mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit} \simeq \mathcal{R}\mathrm{ing}_{B\otimes\Lambda}^{\heartsuit}$ as in Example 3.8, so that $B/\mathcal{R}\mathrm{ing}_{\Lambda}^{\heartsuit}/B \simeq \mathcal{R}\mathrm{ing}_{B\otimes\Lambda}^{\mathrm{aug},\heartsuit}$. Then as a $\mathcal{P}$-algebra, $\Delta_B(B \otimes \Lambda) \cong B \otimes \Delta(\Lambda)$ is a composition of the monad $B \otimes -$ with $\Delta(\Lambda)$. Moreover, the diagram

$$\begin{tikzcd}
\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit/B) \arrow[r] \arrow[d] & \mathcal{A}\mathrm{b}(\mathcal{C}\mathcal{R}\mathrm{ing}_\mathcal{P}^\heartsuit/B) \arrow[d] \\
\mathcal{R}\mathrm{ing}_\Lambda^\heartsuit/B \arrow[r] & \mathcal{C}\mathcal{R}\mathrm{ing}_\mathcal{P}^\heartsuit/B
\end{tikzcd}$$

is distributive.

PROOF. Given Lemma 3.18, these are just specializations of the general theory of Subsection 3.1.9. □

REMARK 3.9. The composite $\Gamma(\Lambda) \to \Lambda \to \Delta(\Lambda)$ is a map of $\mathcal{P}$-algebras. ◁

REMARK 3.10. The constructions of $\Gamma(\Lambda)$ and $\Delta(\Lambda)$ are formally dual: if we additively split $\Lambda_P = \widetilde{\Lambda}_P \oplus \mathbb{1}$, then

$$\Delta(\Lambda)_P = \mathrm{Coker}(\widetilde{\Lambda}_{P\oplus P} \to \widetilde{\Lambda}_P)$$
$$\Gamma(\Lambda)_P = \mathrm{Ker}(\widetilde{\Lambda}_P \to \widetilde{\Lambda}_{P\oplus P}),$$

the maps being obtained from the codiagonal $P \oplus P \to P$ and diagonal $P \to P \oplus P$ respectively. In other words, $\Delta(\Lambda)$ is the linearization of the functor Λ, and dually we might call $\Gamma(\Lambda)$ the colinearization of Λ. ◁

EXAMPLE 3.19. Let Λ be the $\mathbb{Z}$-plethory of θ-rings, also known as δ-rings [**Joy85**], as well as by other names. In brief, θ-rings are commutative rings B equipped with an operation $\theta\colon B \to B$ satisfying all the identities necessary to make

$$\psi(b) = b^p + p\theta(b)$$

generically a ring map. The underlying commutative ring of Λ can be identified as

$$\Lambda = \mathbb{Z}[\theta_n : n \geq 0],$$

where $\theta_n = \theta^{\circ n}$. Here, we are making use of the correspondence between elements of the ring Λ and natural operations on θ-rings, so for example the operation ψ is given by the element

$$\psi = \theta_0^p + p\theta_1.$$

The operation ψ freely generates the additive operations, and we can identify $\Gamma(\Lambda) = \mathbb{Z}[\psi]$ as a $\mathbb{Z}$-algebra. As ψ is a ring homomorphism, the cobialgebroid structure is

$$\epsilon^\times(\psi) = 1, \qquad \Delta^\times(\psi) = \psi \otimes \psi.$$

Evidently $\Delta(\Lambda) = \mathbb{Z}[\theta]$ as a $\mathbb{Z}$-algebra, and the map $\Gamma(\Lambda) \to \Delta(\Lambda)$ is given by

$$\mathbb{Z}[\psi] \to \mathbb{Z}[\theta], \qquad \psi \mapsto p\theta.$$

Now say B is a θ-ring, so that $B \otimes \Lambda$ is a B-plethory. The general recipe of Example 3.8 for computing the plethory structure on $B \otimes \Lambda$ translates into the following. Note $B \otimes \Lambda = B[\theta_n : n \geq 0]$, and it is sufficient to determine the composition $\theta_1 \circ b$ for $b \in B$. As an element of $B \otimes \Lambda$, the element $\theta_1 \circ b$ represents the natural operation

$$(\theta_1 \circ b)(a) = \theta(b \cdot a) = \theta(b)a^p + b^p\theta(a) + p\theta(b)\theta(a),$$

defined for θ-rings under B. Thus we have

$$\theta_1 \circ b = \theta_1(b)\theta_0^p + b^p\theta_1 + p\theta(b)\theta_1.$$

This is nothing but a specialization of the general formula

$$(b \otimes \sigma) \circ (b' \otimes \sigma') = \sum b\sigma^{\times}_{(1)}(b') \otimes \sigma^{\times}_{(2)} \circ \sigma'.$$

We can identify $\Gamma(B \otimes \Lambda) = B \otimes \Gamma(\Lambda) = B[\psi]$ as $\mathbb{Z}$-bimodules. For clarity, write $\psi_1 = \psi$ as an element of $B \otimes \Gamma(\Lambda)$. The algebra structure is determined by the general distributive law of Remark 3.7, which is in turn described by Example 3.7, and thus

$$\Gamma(B \otimes \Lambda) = B\langle\psi_1\rangle/(\psi_1 \cdot b = \psi(b) \cdot \psi_1)$$

as a B-algebra. Similarly $\Delta(B \otimes \Lambda) = B \otimes \Delta(\Lambda) = B[\theta]$ as $\mathbb{Z}$-bimodules, but as $\Delta(\Lambda)$ does not carry a coproduct, the algebra structure cannot be determined in the same way. There are two good ways to proceed. The method that works in general is to observe that the composition law on $B \otimes \Lambda$ implies that $\theta \circ b \equiv \psi(b)\theta$ mod indecomposables, and thus

$$B \otimes \Delta(\Lambda) = B\langle\theta\rangle/(\theta \cdot b = \psi(b) \cdot \theta).$$

The second method is to observe that the existence of an algebra map

$$B\langle\psi_1\rangle/(\psi_1 \cdot b = \psi(b) \cdot \psi_1) \to B \otimes \mathbb{Z}[\theta], \qquad \psi_1 \mapsto p\theta$$

determines the algebra structure on $B \otimes \mathbb{Z}[\theta]$, at least when B is p-torsion free. $\triangleleft$

Call Λ *smooth* when its associated $S\mathcal{P}$-algebra is smooth in the sense of Subsection 3.1.9; that is, if $\Lambda(P) \in \mathcal{CR}\mathrm{ing}_{\mathcal{P}}$ is smooth for all $P \in \mathcal{P}$. When Λ is smooth, Proposition 3.4 and Proposition 3.5 allow one to split computations of the Quillen cohomology of Λ-rings into computations of the Quillen cohomology of $\mathcal{P}$-rings together with computations of Ext over the additive $\mathcal{P}$-algebra $\Delta(\Lambda)$.

It is convenient to have a relative version of this. Given a map $\Lambda' \to \Lambda$ of $\mathcal{P}$-plethories, Λ may be viewed as an algebra for the theory of Λ'-rings, and we say that Λ *smooth relative to* Λ' when it is smooth in this sense.

Example 3.20. Let R be an ordinary commutative ring, and $\mathcal{P}$ the theory of $\mathbb{Z}$-graded R-modules, regarded as a symmetric monoidal theory with symmetrizer employing the Koszul sign rule. Then the category $\mathrm{C}\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit} \simeq \mathrm{C}\mathcal{M}\mathrm{on}(\mathcal{M}\mathrm{od}_{R_*}^{\heartsuit})$ may be idntified as the category of ordinary $\mathbb{Z}$-graded R-algebras B such that

$$bb' = (-1)^{|b||b'|}b'b$$

for all $b, b' \in B$. In particular, if we write $R\{e_n\}$ for a copy of R in degree n, then

$$SR\{e_n\} = R[e_n]/((1-(-1)^n)e_n^2).$$

When 2 is neither zero nor a unit in R, the monad S need not preserve projective objects, and the homotopy theory of simplicial R_*-rings may not behave as one would like. One fix is to instead work with *alternating R_*-algebras*, i.e. those $B \in \mathrm{C}\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$ such that $b^2 = 0$ when $|b|$ is odd. Denote this category by $\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$. Then the inclusion $\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit} \to \mathrm{C}\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$ is plethystic, and moreover $\mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}/B) \simeq \mathcal{M}\mathrm{od}_{B_*}^{\heartsuit} \simeq \mathcal{A}\mathrm{b}(\mathrm{C}\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}/B)$, so everything we have done for R_*-plethories carries over verbatim to the relative setting over $\mathcal{R}\mathrm{ing}_{R_*}^{\heartsuit}$. ◁

3.3.5. Suspension maps. We record here a definition of an additional piece of structure present in plethories that encode homotopy operations. Fix a symmetric monoidal additive theory $\mathcal{P}$ as before. Let E be an object of $\mathcal{P}$ which is invertible under the tensor product. Then there is an automorphism of the category of endofunctors of $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ given by

$$H \mapsto H^E, \qquad H^E(M) = E \otimes H(E^{-1} \otimes M).$$

This is compatible with compositions of endofunctors, and so preserves monads and comonads. In addition, it preserves bimodules, and $(H^E)^\vee = (H^\vee)^{E^{-1}}$.

Definition 3.11.

(1) If F is a $\mathcal{P}$-algebra, we say that F is equipped with *E-suspensions* if we have chosen a map $\sigma\colon F^E \to F$ of algebras.

(2) If Γ is a $\mathcal{P}$-bialgebroid, we say that Γ is equipped with *E-suspensions* if we have equipped the underlying algebra of Γ with E-suspensions in such a way that for all $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}^{\heartsuit}$, the diagram

$$\begin{tikzcd}
\Gamma^E(M) \arrow[rrr] \arrow[d, "="] & & & \Gamma(M) \\
E \otimes \Gamma(E^{-1} \otimes M) \arrow[r, "E\otimes\Delta^\times"] & E \otimes \Gamma(E^{-1}) \otimes \Gamma(M) \arrow[rr, "\sigma\otimes\Gamma(M)"] & & \Gamma(\mathbb{1}) \otimes \Gamma(M) \arrow[u, "\epsilon^\times\otimes\Gamma(M)"']
\end{tikzcd}$$

commutes.

(3) If Λ is a $\mathcal{P}$-plethory, we say that Λ is equipped with *E-suspensions* if we have chosen a map $\sigma\colon \Delta(\Lambda)^E \to \Gamma(\Lambda)$ of algebras such that the composite $\Gamma(\Lambda)^E \to \Delta(\Lambda)^E \to \Gamma(\Lambda)$ equips $\Gamma(\Lambda)$ with E-suspensions. ◁

This definition is not intended to cover all cases where one may wish to speak of suspension maps, but only those we will need in Chapter 4. In all of our explicit examples, $\mathrm{LMod}_{\mathcal{P}}$ will be a category of $\mathbb{Z}$-graded objects. Here we will always take E to be a copy of the monoidal unit in degree 1, and will just say "equipped with suspensions", as we trust no confusion should arise.

Example 3.21. Let k be an ordinary commutative ring, and consider the theory of $\mathbb{Z}$-graded left k-modules. Let E denote a copy of k in degree 1. If B is an ordinary $\mathbb{Z}$-graded k-algebra, then underlying monad of B is equipped with E-suspensions given by the identifications

$$E \otimes B \otimes E^{-1} \otimes M = B \otimes M.$$

As remarked in Example 3.12, when B is augmented, this is the sort of structure needed to define $H^*(B)$ as an ordinary $\mathbb{Z}$-graded algebra. $\triangleleft$

Given an object T equipped with a suspension map $\sigma\colon T^E \to T$, we can define the *costabilization* of T to be $\lim_{n\to\infty} T^{E^n}$. This is a monad when σ is a map of monads.

Example 3.22. Let $\mathcal{U}$ be the algebra for the theory $\mathbb{Z}$-graded $\mathbb{F}_2$-modules whose modules are the unstable modules over the mod 2 Steenrod algebra. Then $\mathcal{U}$ is naturally equipped with suspensions $\sigma\colon \mathcal{U}^E \to \mathcal{U}$ given by $\sigma(\mathrm{Sq}^I) = \mathrm{Sq}^I$, with the understanding that this element may be zero in the target even when nonzero in the source. The costabilization of $\mathcal{U}$ is isomorphic to the mod 2 Steenrod algebra. $\triangleleft$

CHAPTER 4

Applications

4.1. $\mathbb{E}_\infty$ rings over $\mathbb{F}_p$

In this section we describe what the content of the preceding chapters looks like in the context of power operations for $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$.

In Subsection 4.1.1, we show how one can, by purely formal considerations, construct a plethory DL encoding the structure of mod p power operations. To go further one needs real knowledge of mod p power operations, and we review the structure of these in Subsection 4.1.2. These DL-rings are very similar to unstable rings over the Steenrod algebra, and we describe the precise relationship in Subsection 4.1.3. The plethory DL is smooth, and so the Quillen cohomology of DL-rings fits into the general story of Subsection 3.3.4; we highlight some features in Subsection 4.1.4. Given this, the Quillen cohomology of DL-rings splits into a classic portion, governed by the ordinary André-Quillen cohomology of graded commutative rings, and a linear portion, governed by the cotangent algebra $\Delta(\mathrm{DL})$. This cotangent algebra turns out to be Koszul, and we compute its cohomology in Subsection 4.1.5.

With all of this algebra in place, we are able to finally give homotopical applications. In Subsection 4.1.6, we unravel what the mapping space obstruction theory of Theorem 2.11 says in this context, and give some general examples. In Subsection 4.1.7, we apply Theorem 2.7 to produce a form of the Basterra spectral sequence for computing topological André-Quillen homology and cohomology in this context.

Throughout this section, we will write e_a for a generic $\mathbb{Z}$-graded module generated by an element in degree a.

4.1.1. Plethories of power operations. To illustrate the relevant ideas, we show how one can identify that a plethory of mod p power operations exists, even before the hard work of computing its structure has been carried out. Our work is simplified by the fact that $\mathbb{F}_p$ is a field, but the approach taken here generalizes to other contexts.

Let $\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ denote the category of $H\mathbb{F}_p$-module spectra; we will just call these $\mathbb{F}_p$-modules. This is a symmetric monoidal category under $\otimes = \otimes_{\mathbb{F}_p}$. Let $\mathbb{P}$ denote the free $\mathbb{E}_\infty$ algebra monad on $\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$,

$$\mathbb{P}V = \bigoplus_{n\geq 0} \mathbb{P}_n V, \qquad \mathbb{P}_n V = V^{\otimes n}_{h\Sigma_n},$$

and $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$ the resulting category of $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$. Let $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}} \subset \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$ denote the essential image of $\mathbb{P}$. Both $\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ and $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$ are theories, and thus so are their homotopy categories $\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ and $\mathrm{h}\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$.

The theory $\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ is easily identified:

$$\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p} \simeq \mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}^{\heartsuit}, \qquad \mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p}} \simeq \mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}.$$

Here $\mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}$ is the (derived) category of $\mathbb{Z}$-graded $\mathbb{F}_p$-modules; we will just call these $\mathbb{F}_{p*}$-modules. Moreover, the symmetric monoidal structure on $\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ obtained from that on $\mathcal{M}\mathrm{od}_{\mathbb{F}_p}$ is exactly the standard symmetric monoidal structure on $\mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}^{\heartsuit}$ with symmetrizer obeying the Koszul sign rule.

By contrast, the theory $\mathrm{h}\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$ is more complicated. Abstractly, one can say that it is exactly the theory of operations acting on the homotopy groups of $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$; for example, following Proposition 3.1, there are isomorphisms

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p},\mathcal{S}\mathrm{et})}(\pi_p,\pi_q) \cong \pi_q\mathbb{P}\Sigma^p\mathbb{F}_p \cong \mathrm{Hom}_{h\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}}(\mathbb{P}\Sigma^q\mathbb{F}_p,\mathbb{P}\Sigma^p\mathbb{F}_p).$$

By construction, $\mathbb{P}$ induces a map $\mathrm{h}\mathcal{M}\mathrm{od}_{\mathbb{F}_p} \to \mathrm{h}\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$ of theories, and restriction along this makes the category of $\mathrm{h}\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$-models strongly monadic over $\mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}$, i.e. we may view $\mathrm{h}\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$-models as $\mathbb{F}_{p*}$-modules equipped with some extra structure. Write DL for the associated monad on $\mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}$; the general properties of this are as outlined in Proposition 2.8. In particular, the following is a consequence of the construction of DL and properties of $\mathbb{P}$.

Proposition 4.1. The natural isomorphisms $\mathbb{P}(U \oplus V) \simeq \mathbb{P}U \otimes \mathbb{P}V$ give natural isomorphisms $\pi_*\mathbb{P}(U \oplus V) \simeq \pi_*\mathbb{P}U \otimes \pi_*\mathbb{P}V$ which equip DL with the structure of an exponential monad on $\mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}^{\heartsuit}$, and thus DL is a $\mathbb{F}_{p*}$-plethory. Moreover, the natural maps $\Sigma\mathbb{P}_n V \to \mathbb{P}_n\Sigma V$ defined for $n \geq 1$ equip DL with suspensions. $\square$

Remark 4.1. As all $\mathbb{F}_p$-modules are free, $\mathcal{R}\mathrm{ing}_{\mathrm{DL}}^{\heartsuit}$ may also be identified as the category of $\mathbb{H}_\infty$ algebras over $\mathbb{F}_p$. $\triangleleft$

This is as far as purely formal considerations can take us; to get further one needs real knowledge of the structure of mod p power operations.

4.1.2. Dyer-Lashof operations. Our goal now is to recall the structure of mod p power operations, and package to it into the plethystic framework. We find it most convenient to proceed by introducing some of the relevant algebra first. We begin by recalling a certain algebra $\mathcal{B}$ of power operations; this algebra has various names in the literature, such as the big, or generalized, Steenrod algebra, or the Kudo-Araki-May algebra.

We follow the convention that the binomial coefficient $\binom{n}{m}$ vanishes unless $0 \leq m \leq n$.

Definition 4.1 ($p = 2$). $\mathcal{B}$ is the ordinary $\mathbb{Z}$-graded associative $\mathbb{F}_2$-algebra generated by symbols Q^s of degree s for all $s \in \mathbb{Z}$, and subject to the relations

$$Q^{2s+r+1}Q^s = \sum_{0 \le i < \frac{r}{2}} \binom{r-i-1}{i} Q^{2s+i+1}Q^{r+s-i}$$

for $r \geq 0$. Here, the bounds of summation are not necessary, but indicate when the binomial coefficients may be nonzero. Given a sequence $I = (r_1, \ldots, r_k)$, write $Q^I = Q^{r_1} \cdots Q^{r_k}$, and call I and Q^I *admissible* if $r_i \leq 2r_{i+1}$ for each i. Define the *excess* of I by $e(I) = r_1 - r_2 - \cdots - r_k$. Given an integer u, call a $\mathcal{B}$-module M *u-unstable* if $Q^r m = 0$ for any $m \in M$ with $r < |m| + u$; when $u = 0$ we omit it from the name and notation. $\triangleleft$

Definition 4.2 ($p > 2$). $\mathcal{B}$ is the ordinary $\mathbb{Z}$-graded associative $\mathbb{F}_p$-algebra generated by symbols Q^s_ϵ of degree $2s(p-1) - \epsilon$ for $\epsilon \in \{0, 1\}$ and $s \in \mathbb{Z}$, and subject to the relations

$$\begin{aligned}
Q^{ps+r+1}Q^s &= \sum_{0 \le i < \frac{p-1}{p}s} (-1)^{i+1} \binom{(p-1)(r-i)-1}{i} Q^{ps+i+1}Q^{r+s-i} \\
Q^{ps+r}Q^s_1 &= \sum_{0 \le i \le \frac{p-1}{p}r} (-1)^{i} \binom{(p-1)(r-i)}{i} Q^{ps+i}_1 Q^{r+s-i} \\
&\quad + \sum_{0 \le i < \frac{p-1}{p}r} (-1)^{i+1} \binom{(p-1)(r-i)-1}{i} Q^{ps+i}Q^{r+s-i}_1 \\
Q^{ps+r+1}_1 Q^s &= \sum_{0 \le i < \frac{p-1}{p}r} (-1)^{i+1} \binom{(p-1)(r-i)-1}{i} Q^{ps+i+1}_1 Q^{r+s-i} \\
Q^{ps+r}_1 Q^s_1 &= \sum_{0 \le i < \frac{p-1}{p}r} (-1)^{i+1} \binom{(p-1)(r-i)-1}{i} Q^{ps+i}_1 Q^{r+s-i}_1
\end{aligned}$$

for $r \geq 0$, where we have abbreviated $Q^s = Q^s_0$. Here, the bounds of summation are not necessary, but indicate when the binomial coefficients may be nonzero. Given a sequence $I = (\epsilon_1, r_1, \ldots, \epsilon_k, r_k)$ in $\{0,1\} \times \mathbb{Z}$, write $Q^I = Q^{r_1}_{\epsilon_1} \cdots Q^{r_k}_{\epsilon_k}$, and call I and Q^I *admissible* if $r_i \leq pr_{i+1} - \epsilon_{i+1}$ for each i. Define the *length* of I to be k and the *excess* of I to be $e(I) = 2r_1 - \epsilon_1 - (2r_2(p-1) - \epsilon_2) - \cdots - (2r_k(p-1) - \epsilon_k)$. Given an integer u, call a $\mathcal{B}$-module M *u-unstable* if $Q^r_\epsilon m = 0$ for any $m \in M$ with $2r - \epsilon < |m| + u$; when $u = 0$, we omit it from the name and notation. $\triangleleft$

For any integer u, write F^u for the free u-unstable $\mathcal{B}$-module functor. Then F^u is a quotient algebra of $\mathcal{B}$, where here we refer to the general notion of algebra studied in Chapter 3, as well as of $F^{u'}$ for $u' < u$. Write $E = e_1$, and for $M \in \mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}$, write $sM = E \otimes M$. If M is an F^u-module, then sM is an F^{u-1}-module, and this provides an isomorphism $(F^u)^E \cong F^{u-1}$; together with the quotient maps $F^{u-1} \to F^u$, this equips each algebra F^u with suspensions, and for the most part reduces us to considering just $F = F^0$.

Lemma 4.1.

(1) $\mathcal{B}$ has a basis consisting of Q^I for all admissible sequences I;
(2) $F(e_n)$ has a basis consisting of $Q^I e_n$ with I admissible of excess at least n.

Proof. This is [**Man01**, Propositions 11.2 and 12.2]; see also [**Lur07**, Lectures 6-7] for a detailed algebraic proof when $p = 2$ which, provided one assumes the analogous fact for unstable modules over the Steenrod algebra [**Sch94**, Proposition 1.6.2], generalizes to $p > 2$. □

Remark 4.2. The abelian category $\mathrm{L}\mathcal{M}\mathrm{od}_F^{\heartsuit}$ has enough projectives, given by the free F-modules. In addition, the right adjoint $F^\vee$ supplies it with enough injectives. By definition,

$$F^\vee(e_a)_b = \mathcal{M}\mathrm{od}_{\mathbb{F}_p}(F(e_b), e_a),$$

so the injective modules $F^\vee(e_a)$ can be seen as analogues of the Brown-Gitler modules seen in the study of unstable modules over the Steenrod algebra. ◁

Definition 4.3. A DL-*ring* is a graded commutative $\mathbb{F}_{p*}$-ring equipped with an F-module structure such that

(1) $Q^0(1) = 1$, and otherwise $Q^r_\epsilon(1) = 0$;
(2) (a) If $p = 2$, then $Q^r x = x^2$ when $r = |x|$,
    (b) If $p > 2$, then $Q^r x = x^p$ when $2r = |x|$;
(3) (a) If $p \geq 2$, then $Q^r(xy) = \sum_{i+j=r} Q^i(x)Q^j(y)$,
    (b) If $p > 2$, then $Q^r_1(xy) = \sum_{i+j=r}(Q^i_1(x)Q^j(y) + (-1)^{|x|}Q^i(x)Q^j_1(y))$. ◁

From the definition, we see that DL-rings are the models of a finite product theory living over $\mathcal{M}\mathrm{od}^{\heartsuit}_{\mathbb{F}_{p*}}$; write the associated monad as DL. Let DL_n denote the free $\mathbb{F}_{p*}$-ring on symbols $Q^I e_n$ where I is an admissible sequence satisfying $e(I) > n$, graded so that $|Q^I e_n| = |Q^I| + n$. The action of $\mathcal{B}$ on the canonical element of $\mathrm{DL}(e_n)_n$ gives a map $\mathrm{DL}_n \to \mathrm{DL}(e_n)$ of $\mathbb{F}_{p*}$-rings.

Theorem 4.1 ([**BMMS86**, III.1.1, IX.2.1]). The structure of mod p power operations can be summarized as follows.

(1) The homotopy groups of any object of $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$ naturally form a DL-ring, and the resulting maps $\mathrm{DL}_n \to \mathrm{DL}(e_n) \to \pi_*\mathbb{P}\Sigma^n\mathbb{F}_p$ are all isomorphisms;
(2) In particular, DL agrees with the $\mathbb{F}_{p*}$-plethory of Subsection 4.1.1, and is a smooth $\mathbb{F}_{p*}$-plethory with suspensions;
(3) There are isomorphisms $\Gamma(\mathrm{DL}) \cong F$ and $\Delta(\mathrm{DL}) \cong F^1$, and the suspension map $\sigma\colon \Delta(\mathrm{DL})^E \to \Gamma(\mathrm{DL})$ is given by the isomorphism $(F^1)^E \cong F$. □

Throughout the rest of this section, we will abbreviate $\Delta = \Delta(\mathrm{DL}) = F^1$.

EXAMPLE 4.1. The costabilization $\lim_{n\to\infty} \Gamma(\mathrm{DL})^{E^n}$ of DL can be identified as the completion of $\mathcal{B}$ with respect to excess. This object arises naturally when considering "stable power operations", as realized by the endomorphism spectrum of the forgetful functor $U\colon \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p} \to \mathcal{S}\mathrm{p}$; see [**Lur07**, Lecture 24] and [**GL20**, Section 10]. $\triangleleft$

4.1.3. Unstable $\mathcal{A}$-modules. Let $\mathcal{A}$ denote the mod p Steenrod algebra. As observed by Mandell [**Man01**, Theorem 1.4], there is a quotient map

$$\mathcal{B} \to \mathcal{B}/(Q^0 = 1) \cong \mathcal{A}, \qquad \begin{cases} Q^s \mapsto \mathrm{Sq}^{-s}, & \text{when } p = 2; \\ Q^s_\epsilon \mapsto \beta^\epsilon P^{-s}, & \text{when } p > 2. \end{cases}$$

DEFINITION 4.4. An *unstable $\mathcal{A}$-module*, resp., *unstable $\mathcal{A}$-ring*, is an F-module, resp., DL-ring, whose underlying $\mathcal{B}$-module structure factors through the quotient map $\mathcal{B} \to \mathcal{A}$. $\triangleleft$

Unstable $\mathcal{A}$-modules and unstable $\mathcal{A}$-rings (more commonly called unstable $\mathcal{A}$-algebras) have been the study of much rich study; see [**Sch94**] for a textbook account, and [**Lur07**] for an account that treats the relation with $\mathcal{B}$. Essentially everything we do with F-modules and DL-rings has an analogue for unstable $\mathcal{A}$-modules and unstable $\mathcal{A}$-rings.

Write $\mathcal{U}$ for the $\mathbb{F}_{p*}$-algebra such that $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{U}}^{\heartsuit}$ is the category of unstable $\mathcal{A}$-modules (itself often written $\mathcal{U}$), as in Example 3.16. Write $\mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\heartsuit}$ for the category of unstable $\mathcal{A}$-rings. By definition, unstable $\mathcal{A}$-rings embed fully faithfully into DL-rings. This is no longer the case at the level of simplicial rings, i.e. the functor $\mathcal{R}\mathrm{ing}_{\mathcal{U}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}}$ is no longer fully faithful. The situation here is exactly the same as that appearing in [**Man01**], and can be dealt with the same way.

Write $T\colon \mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}} \to \mathcal{R}\mathrm{ing}_{\mathcal{U}}$ for the free unstable $\mathcal{A}$-ring functor.

LEMMA 4.2. For all $n \in \mathbb{Z}$, there is a (homotopy) pushout square

$$\begin{tikzcd} \mathrm{DL}(e_n) \arrow[r, "\phi"] \arrow[d] & \mathrm{DL}(e_n) \arrow[d] \\ \mathbb{F}_p \arrow[r] & T(e_n) \end{tikzcd}$$

in $\mathcal{R}\mathrm{ing}_{\mathrm{DL}}$, where ϕ classifies the element $e_n - Q^0 e_n$.

PROOF. See [**Man01**, Section 12]. $\square$

If R is a discrete commutative $\mathbb{F}_p$-ring, then R can be viewed as an $\mathbb{E}_\infty$ algebra over $\mathbb{F}_p$. The resulting DL-ring structure on $R = \pi_* R$ is forced by the axioms to satisfy $Q^0 x = x^p$ and otherwise $Q^r_\epsilon x = 0$.

Call a field κ of characteristic p *Artin-Schreier closed* if the map $\lambda \mapsto \lambda - \lambda^p$ is surjective on κ. In particular, this holds if κ is algebraically closed.

Proposition 4.2. Let κ be an Artin-Schreier closed field. Then the composite

$$\mathcal{R}\mathrm{ing}_{\mathcal{U}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}} \to \mathcal{R}\mathrm{ing}_{\kappa\otimes\mathrm{DL}}$$

is fully faithful.

Proof. We first verify this on discrete objects. Fix $R, S \in \mathcal{R}\mathrm{ing}_{\mathcal{U}}^{\heartsuit}$. Then $\mathrm{Hom}_{\mathcal{U}}(R,S) = \mathrm{Hom}_{\mathrm{DL}}(R,S)$, and we must show this is isomorphic to $\mathrm{Hom}_{\kappa\otimes\mathrm{DL}}(\kappa\otimes R, \kappa\otimes S)$. The latter is isomorphic to $\mathrm{Hom}_{\mathrm{DL}}(R, \kappa\otimes S)$ by adjunction, so we must show that every map $f\colon R \to \kappa\otimes S$ of DL-rings factors through $\mathbb{F}_p \otimes S \subset \kappa\otimes S$. As Q^0 acts by the identity on R, every such map lands in the fixed points of Q^0 on $\kappa\otimes S$. As Q^0 acts on $\kappa\otimes S$ by $Q^0(\lambda\otimes s) = \lambda^p \otimes s$, the set of fixed points of Q^0 on $\kappa\otimes S$ is exactly $\mathbb{F}_p\otimes S$, proving the claim.

Now fix $R, S \in \mathcal{R}\mathrm{ing}_{\mathcal{U}}$ which are not necessarily discrete, and consider the map

$$\mathrm{Map}_{\mathcal{U}}(R,S) \to \mathrm{Map}_{\mathrm{DL}}(R, \kappa\otimes S)$$

which we are claiming is an isomorphism. As $\mathcal{R}\mathrm{ing}_{\mathcal{U}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}}$ is stable under colimits, we may by resolving R reduce to the case where $R = T(e_n)$ is a free unstable $\mathcal{A}$-ring, and so reduce to verifying that the map

$$S_n = \mathrm{Map}_{\mathcal{U}}(T(e_n), S) \to \mathrm{Map}_{\mathrm{DL}}(T(e_n), \kappa\otimes S)$$

is an isomorphism. By Lemma 4.2, there is a fiber sequence

$$\mathrm{Map}_{\mathrm{DL}}(T(e_n), \kappa\otimes S) \to \kappa\otimes S_n \to \kappa\otimes S_n,$$

where the second map is given on homotopy groups by $\lambda\otimes s \mapsto (\lambda - \lambda^p)\otimes s$. The claim follows from the long exact sequence in homotopy groups. $\square$

4.1.4. Cohomology of DL-rings. Because DL is a smooth $\mathbb{F}_{p*}$-plethory, the general aspects of the cohomology of DL-rings is as described in Subsection 3.3.4. Explicitly, fix $R \in \mathcal{R}\mathrm{ing}_{\mathrm{DL}}^{\heartsuit}$, $B \in \mathcal{R}\mathrm{ing}_{R\otimes\mathrm{DL}}^{\heartsuit}$, $A \in \mathcal{R}\mathrm{ing}_{R\otimes\mathrm{DL}/B}^{\heartsuit}$, and $M \in \mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{R\otimes\mathrm{DL}/B}^{\heartsuit}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{B\otimes\Delta}^{\heartsuit}$. Here, if $p = 2$, then $B\otimes\Delta$ has multiplication given by

$$(b\otimes Q^r)\cdot(b'\otimes Q^{r'}) = \sum_{i+j=r} b\, Q^i(b')\otimes Q^j Q^{r'},$$

and if $p > 2$, by

$$(b\otimes Q_1^r)\cdot(b'\otimes Q^{r'}) = \sum_{i+j=r}\left(b\, Q_1^i(b')\otimes Q^j Q^{r'} + (-1)^{|b'|} b\, Q^i(b')\otimes Q_1^j Q^{r'}\right);$$

these follow from the recipe of Theorem 3.6. By smoothness, $\mathbb{L}\Omega_{A|R}$ upgrades to an $A\otimes\Delta$-module, and

$$\mathcal{H}_{R\otimes\mathrm{DL}/B}(A; M) \simeq \mathcal{E}\mathrm{xt}_{B\otimes\Delta}(B\otimes_A^{\mathbb{L}} \mathbb{L}\Omega_{A|R}, M),$$

In particular, if A is smooth over R, then $H^*_{R\otimes \mathrm{DL}/B}(A;M) = \mathrm{Ext}^*_{B\otimes\Delta}(B\otimes_A \mathbb{L}\Omega_{A|R}, M)$.

There is a complementary method by which these Quillen cohomology computations can, in certain cases, be reduced to linear computations. Observe that the forgetful functor $\mathcal{R}\mathrm{ing}^{\heartsuit}_{\mathrm{DL}} \to \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_F$ admits a left adjoint S_F, described by

$$S_F M = \begin{cases} SM/(Q^r x = x^2 \text{ for } r = |x|), & \text{for } p = 2; \\ SM/(Q^r x = x^p \text{ for } 2r = |x|), & \text{for } p > 2. \end{cases}$$

In fact this is already derived, i.e. agrees with its total derived functor on discrete objects, as can be seen from the description

$$S_F M = SM\ {}_{\psi}\!\otimes_{S\Psi M} \mathbb{F}_p,$$

where

$$(\Psi M)_n = \begin{cases} M_{n/p}, & \text{when } 2, p | n; \\ 0, & \text{otherwise;} \end{cases} \qquad \psi(m) = \begin{cases} m^2 - Q^{|m|}m, & \text{for } p = 2; \\ m^p - Q^{|m|/2}m, & \text{for } p > 2. \end{cases}$$

As a consequence, if $M \in \mathrm{L}\mathcal{M}\mathrm{od}^{\heartsuit}_F$ and $B \in \mathcal{R}\mathrm{ing}_{\mathrm{DL}}$, then $\mathrm{Map}_{\mathrm{DL}}(S_F M, B) \simeq \mathrm{Map}_F(M, B)$, and this provides an approach to computing the cohomology of DL-rings in the image of S_F.

These constructions easily extend to bases other than $\mathbb{F}_p$ and to augmented settings. In particular, if $M \in \mathrm{L}\mathcal{M}\mathrm{od}_F$ and $N \in \mathrm{L}\mathcal{M}\mathrm{od}_\Delta$, then

$$\mathcal{H}_{\mathrm{DL}/\mathbb{F}_p}(S_F M; N) \simeq \mathcal{E}\mathrm{xt}_F(M, N).$$

Example 4.2. The module e_n always carries an F-module structure where each Q^r_ϵ acts by zero; this is the action obtained from the augmentation on F. If $n \geq 0$, then e_{-n} carries a second F-module structure, where Q^0 acts by the identity. If e'_{-n} refers to this F-module structure, then $S_F(e'_{-n})$ is isomorphic to the cohomology algebra of the n-sphere, where if n is even and p is odd then we must take the "homotopy theorist's even sphere" $J_{p-1}S^n$ [**Gra93**]. ◁

4.1.5. The big lambda algebra. We turn now to the construction of Koszul complexes computing Ext_F. This discussion applies equally well to Ext_Δ, or to Ext_{F^u} for $u \in \mathbb{Z}$, as well to $\mathrm{Ext}_{\mathcal{U}}$ (Proposition 4.3). Moreover, by Lemma 3.12, it extends to $\mathrm{Ext}_{B\otimes F}$ for $B \in \mathcal{R}\mathrm{ing}^{\heartsuit}_F$, and to related contexts. These Koszul complexes have a number of predecessors, particularly with work of Miller [**Mil78**] in the connective setting, and with the unstable lambda complexes seen in work on the unstable Adams spectral sequence [**BK72b**]. We have found that working in the full $\mathbb{Z}$-graded setting serves to clarify some of the algebra.

We begin by observing that F is a quadratic algebra. If we write its length grading as $F = \bigoplus_{n\geq 0} F[n]$, then the generating bimodule is

$$F[1](e_a) = \begin{cases} \mathbb{F}_2\{Q^r e_a : r \geq a\}, & \text{when } p = 2; \\ \mathbb{F}_p\{Q^r_\epsilon e_a : 2r - \epsilon \geq a\}, & \text{when } p > 2. \end{cases}$$

The relations are just the image of the relations defining $\mathcal{B}$ under the projection $\mathcal{B}[1]\otimes\mathcal{B}[1] \to F[1] \circ F[1]$.

Lemma 4.3. The algebra F is Koszul over $\mathbb{F}_{p*}$.

Proof. The algebra F is locally finite, and its admissible basis may be viewed as a PBW decomposition, so Proposition 3.8 applies. □

Thus there is indeed a theory of Koszul resolutions for computing Ext_F, which we gain access to as soon as we understand the cohomology of F.

We will describe the cohomology of F in two ways, each shedding light on different aspects of the computation. The first approach proceeds by comparing the cohomology of F with the cohomology of the ordinary algebra $\mathcal{B}$, and the second approach proceeds by directly applying Theorem 3.2.

The first approach is plausible due to the following.

Lemma 4.4. The surjection $\mathcal{B} \to F$ yields an injection $H^*(F) \subset H^*(\mathcal{B})$.

Proof. It is sufficient to show dually that $H_*(\mathcal{B}) \to H_*(F)$ is a surjection. As F is Koszul, we need consider only the map on diagonal cohomology. This is given in degree m by

$$\bigcap_{i+j=m} B[1]^{\otimes i-1} \otimes R \otimes B[1]^{\otimes j-1} \to \bigcap_{i+j=m} F[1]^{\circ i-1} \circ R' \circ F[1]^{\circ j-1},$$

where $R \subset B[1] \otimes B[1]$ is the bimodule of Adem relations and $R' \subset F[1] \circ F[1]$ is its image, so this is clear. □

And it is appealing due to the following.

Lemma 4.5. Let D be the diagonal cohomology algebra of $\mathcal{B}$, defined with conventions as in [**Pri70**] (cf. Example 3.11). Let $\hat{Q}^r_\epsilon \in D[1]$ be dual to Q^r_ϵ. Then there is an injection $\mathcal{B} \to D$ of algebras, with dense image, given by

$$\begin{cases} Q^r \mapsto \hat{Q}^{-r-1}, & \text{when } p = 2; \\ Q^r_\epsilon \mapsto \hat{Q}^{-r}_{1-\epsilon}, & \text{when } p > 2. \end{cases}$$

Proof. Though not quite stated in this form, [**Pri70**, Theorem 2.5] gives generators and relations for the diagonal cohomology of an arbitrary ordinary quadratic algebra over a field, with the caveat that if the algebra in question is not locally finite, then these generators may only be topological generators, and the relations obtained may involve infinite sums. In the case of $\mathcal{B}$, it follows by direct computation that the relations obtained between the topological generators $\hat{Q}^r_\epsilon \in D$ are finite, and after the indicated change of indices are exactly the relations defining $\mathcal{B}$. $\square$

From here it is not difficult to proceed to fully describe the cohomology of F. We first lay out some conventions. In the present setting, it is best to compute the cohomology of F with conventions that are standard when dealing with $\mathbb{Z}$-graded modules, only with pairings opposite to Yoneda composition. So for $x \in \mathrm{Ext}^n(e_a, e_b)$ and $y \in \mathrm{Ext}^m(e_b, e_c)$, write

$$xy = (-1)^{n(b-c+m)} y \circ x,$$

where $\circ$ is the Yoneda composition of extensions. With this choice, our pairings are compatible with the graded opposite of [**Pri70**], as discussed in Example 3.11.

Now if we let $\lambda_r \in \mathrm{Ext}^1_{\mathcal{B}}(e_a, e_{a-r-1})$ be the image of Q^r under Lemma 4.5 when $p = 2$, and $\lambda^\epsilon_r \in \mathrm{Ext}^1_{\mathcal{B}}(e_a, e_{a-2r(p-1)+\epsilon-1})$ be the image of Q^r_ϵ under Lemma 4.5 when $p > 2$, then the multiplicative relations between the λ's are exactly the relations in the graded opposite $\mathcal{B}^{\mathrm{op}}$. In particular, the subspace of $\mathrm{Ext}^n_{\mathcal{B}}(e_a, e_{a-*})$ generated under finite sums by products of the λ's is isomorphic, up to shifts in degree, to $\mathcal{B}^{\mathrm{op}}[n]$, and so has basis given by elements λ_I where, if $p = 2$, then $I = (r_1, \ldots, r_n)$ with $2r_i \geq r_{i+1}$ for each i, and if $p > 2$, then $I = (r_1, \epsilon_1, \ldots, r_n, \epsilon_n)$ with $pr_i - \epsilon_i \geq r_{i+1}$ for each i; call these *coadmissible sequences*.

Observe that as F is locally finite, the inclusion $\mathrm{Ext}^n_F(e_a, e_{a-*}) \subset \mathrm{Ext}^n_{\mathcal{B}}(e_a, e_{a-*})$ lands in the subspace isomorphic to $\mathcal{B}^{\mathrm{op}}[n]$ generated under finite sums by the elements λ_I. We have now all but proved the following.

THEOREM 4.2. With the above notation, $\mathrm{Ext}^n_F(e_a, e_{a-*}) \subset \mathrm{Ext}^n_{\mathcal{B}}(e_a, e_{a-*})$ has as basis those λ_I where I is a coadmissible sequence satisfying:

(1) If $p = 2$, then $I = (r_1, \ldots, r_n)$ with $r_1 < -a$;
(2) If $p > 2$, then $I = (r_1, \epsilon_1, \ldots, r_n, \epsilon_n)$ with $2r_1 - \epsilon_1 < -a$.

PROOF. As F has both generators and admissible basis induced by those of $\mathcal{B}$, the functor F may be viewed as a subfunctor of $\mathcal{B}$, though not as a subalgebra. Using Lemma 4.4, it is seen that $\mathrm{Ext}^n_F(e_a, e_{a-*}) \subset \mathrm{Ext}^n_{\mathcal{B}}(e_a, e_{a-*})$ has image spanned by those coadmissible λ_I which lift to elements of $\mathrm{Hom}_{\mathbb{F}_p}(\mathcal{B}[1]^{\otimes n} e_a, e_{a-*})$ dual to simple tensors in $F[1]^{\circ n} e_a$.

(1) Consider the case $p = 2$. Choose a coadmissible sequence $I = (r_1, \ldots, r_n)$. Then we must determine when $\lambda_I \in \mathrm{Hom}_{\mathbb{F}_p}(\mathcal{B}[1]^{\otimes n} e_a, e_{a-*})$ is dual to an element of $F[1]^{\circ n} e_a$. By definition, this element is dual to $Q^{-r_n-1} \otimes \cdots \otimes Q^{-r_1-1}$, and for this to be an element of

$F[1]^{\circ n}e_a$ the sequence I must satisfy the instability condition

$$-r_{i+1}-1 \geq (-r_i-1)+\cdots+(-r_1-1)+a$$

for each i. Write $I' = (s_1,\ldots,s_n) = (-r_n-1,\ldots,-r_1-1)$, so that this instability condition is

$$s_i \geq s_{i+1}+\cdots+s_n+a$$

for each i. Coadmissibility of I is equivalent to the complete unadmissibility condition on I' of $s_i > 2s_{i+1}$ for each i; thus if $s_i \geq s_{i+1}+\cdots+s_n+a$ for some i, then

$$s_{i-1} \geq 2s_i = s_i+s_i \geq s_i+s_{i-1}+\cdots+s_n+a.$$

So in fact the instability condition is equivalent to just $s_n \geq a$, which itself is equivalent to $r_1 < -a$ as claimed.

(2) Consider the case $p>2$. Choose a coadmissible sequence $I=(r_1,\epsilon_1,\ldots,r_n,\epsilon_n)$, so that we must determine when $Q^{-r_n}_{1-\epsilon_n}\otimes\cdots\otimes Q^{-r_1}_{1-\epsilon_1}$ is an element of $F[1]^{\circ n}e_a$. Writing $I'=(\delta_1,s_1,\ldots,\delta_n,\epsilon_n)=(1-\epsilon_n,-r_n,\ldots,1-\epsilon_1,-r_1)$, the relevant instability condition is

$$2s_i-\delta_i \geq (2s_{i+1}(p-1)-\delta_{i+1})+\cdots+(2s_k(p-1)-\delta_k)+a$$

for each i. Coadmissibility of I is equivalent to the complete unadmissibility condition on I' of $s_i > ps_{i+1}-\delta_{i+1}$ for each i; thus if the above is satisfied for some i then

$$2s_{i-1}-\delta_{i-1} > 2(ps_i-\delta_i)-\delta_{i-1} \geq 2s_i(p-1)-\delta_i+\cdots+2s_n(p-1)-\delta_n+a-\delta_{i-1},$$

which in turn implies the instability condition at $i-1$ as $\delta_{i-1}\in\{0,1\}$. So the instability condition is equivalent to just $2s_n-\delta_n\geq a$, which in turn is equivalent to $2r_1-\epsilon_1<-a$. $\square$

The second approach to $H^*(F)$ is through the following.

Theorem 4.3. For $a\in\mathbb{Z}$, the graded vector space $H^*(F)(e_a)$ has basis given by those λ_I where I is a coadmissible sequence satisfying

(1) If $p=2$, then $I=(r_1,\ldots,r_n)$ with $2r_1+r_2+\cdots+r_n+n<-a$;
(2) If $p>2$, then $I=(r_1,\epsilon_1,\ldots,r_n,\epsilon_n)$ with $2(pr_1-\epsilon_1)+2r_2(p-1)-\epsilon_2+\cdots+2r_n(p-1)-\epsilon_n+n<-a$.

Proof. By Theorem 3.2 and Lemma 4.3, there is an isomorphism $H^*(F)=\widehat{T}(F[1]^\vee,R^\perp)$ where $R\subset F[1]\circ F[1]$ is the image of the Adem relations. Here, to maintain consistency with the sign conventions set out above, we should replace $R^\perp$ with $(1\otimes t)(R^\perp)$ where $t(x)=(-1)^{|x|}x$. Note

$$F[1]^\vee(e_a)_b = \mathcal{M}\mathrm{od}_{\mathbb{F}_{p*}}(F[1](e_b),e_a).$$

As before, when $p = 2$, write λ_r for the element of $F[1]^\vee(e_a)$ dual to Q^{-r-1}, with the understanding that this does not exist for all a, and when $p > 2$, write λ_r^ϵ for the element of $F[1]^\vee(e_a)$ dual to $Q_{1-\epsilon}^{-r}$, with the same understanding. Either by appealing to Lemma 4.5, or else by carrying out the same sorts of computations, we find that $R^\perp$ is exactly the space of relations opposite to the Adem relations, only restricted to those elements which live in $F[1]^\vee$. Thus $H^*(F)(e_a)$ has basis given by those λ_I where I is coadmissible and λ_I lifts to $F[1]^{\vee\circ n}(e_a)$.

(1) Consider the case $p = 2$. Here $F[1]^\vee(e_a)_b$ contains an element $\hat{Q}^{a-b}$ dual to Q^{a-b} whenever $a - b \geq b$, i.e. $2b \leq a$. So $F[1]^\vee(e_a)$ is the space of $\hat{Q}^r$ with $2r \geq a$, and in general $F[1]^{\vee\circ n}(e_a)$ is the space of $\hat{Q}^{r_1} \otimes \cdots \otimes \hat{Q}^{r_n}$ with $2r_i + r_{i+1} + \cdots + r_n \geq a$ for each i. This is the space of $\lambda_{r_1} \otimes \cdots \otimes \lambda_{r_n}$ with $2r_i + r_{i+1} + \cdots + n - i + 1 < -a$ for each i. Coadmissibility of I reduces this condition to the case $i = 1$, which is the condition claimed.

(2) Consider the case $p > 2$. Here $F[1]^\vee(e_a)$ contains an element $\hat{Q}_\epsilon^r$ dual to Q_ϵ^r when $2(pr - \epsilon) \geq a$, so in general $F[1]^{\vee\circ n}(e_a)$ consists of those elements $\hat{Q}_{\epsilon_1}^{r_1} \otimes \cdots \otimes \hat{Q}_{\epsilon_n}^{r_n}$ with $2(pr_i - \epsilon_i) + 2r_{i+1}(p-1) - \epsilon_{i+1} + \cdots + 2r_n(p-1) - \epsilon_n \geq a$ for each i. These are the elements $\lambda_{r_1}^{\epsilon_1} \otimes \cdots \otimes \lambda_{r_n}^{\epsilon_n}$ with $2(pr_i - \epsilon_i) + 2r_{i+1}(p-1) - \epsilon_{i+1} + \cdots + 2r_n(p-1) - \epsilon_n + n - i + 1 < -a$ for each i, and again coadmissibility allows one to reduce to the case $i = 1$, which is the condition claimed. $\square$

It is not difficult to directly translate between Theorem 4.2 and Theorem 4.3. In effect, the first describes $\mathrm{Ext}_F^*(e_a, e_*)$, whereas the second describes $\mathrm{Ext}_F^*(e_*, e_a)$.

Remark 4.3. In the preceding, we have avoided the subtle point that though $\mathcal{B}$ has a PBW basis, we cannot apply Proposition 3.8 to deduce that it is Koszul, as the necessary finiteness conditions are not obviously satisfied. Nonetheless $\mathcal{B}$ is Koszul; this was considered in [**BC07**], and we can give an alternate proof as follows. Let H_u be the subalgebra, in our general sense, of $\mathcal{B}^{\mathrm{op}}$, defined so that $H_u(e_a) = H^*(F_u)(e_a) = H^*(F)(e_{a+u})$ is as computed in Theorem 4.3, up to the necessary shifts in degree. Then H_u is a locally finite quadratic algebra admitting a PBW decomposition, so is Koszul. That $\mathcal{B}^{\mathrm{op}}$, and thus $\mathcal{B}$, is Koszul follows from the further observation that $\mathcal{B}^{\mathrm{op}} \cong \mathrm{colim}_{u\to-\infty} H_u$. $\triangleleft$

We end by noting the following, previously mentioned in Example 3.16. We omit the proof, as one may proceed exactly as in the above.

Proposition 4.3. The unstable Steenrod algebra $\mathcal{U}$ is Koszul with respect to its length filtration. Moreover, $\mathrm{gr}\,\mathcal{U} \simeq F/(Q^r : r \geq 0)$, and under this $\mathrm{Ext}^n_{\mathrm{gr}\,\mathcal{U}}(e_a, e_b)$ is isomorphic to the subspace of $\mathrm{Ext}^n_F(e_a, e_b)$ spanned by those λ_I where I is a sequence of nonnegative integers. $\square$

4.1.6. Mapping spaces. We now give applications to the homotopy theory of $\mathbb{E}_\infty$ algebras over $\mathbb{F}_p$.

THEOREM 4.4. Fix $R \in \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$, and choose $S \in \mathcal{C}\mathrm{Alg}_R$ such that $R_* \to S_*$ is surjective (such as $S = 0$ or $S = R$). Choose $A, B \in \mathcal{C}\mathrm{Alg}_{R/S}$, and fix a map $\phi\colon A_* \to B_*$ in $\mathcal{R}\mathrm{ing}_{R_*\otimes \mathrm{DL}/S_*}$. Let $\mathcal{C}\mathrm{Alg}_{R/S}^\phi(A,B)$ be the space of lifts of ϕ to a map in $\mathcal{C}\mathrm{Alg}_{R/S}$. Then there is a decomposition

$$\mathcal{C}\mathrm{Alg}_{R/S}^\phi(A,B) \simeq \lim_{n\to\infty} \mathcal{C}\mathrm{Alg}_{R/S}^{\phi,\leq n}(A,B),$$

with layers fitting into fiber sequences

$$\mathcal{C}\mathrm{Alg}_{R/S}^{\phi,\leq n}(A,B) \to \mathcal{C}\mathrm{Alg}_{R/S}^{\phi,\leq n-1}(A,B) \to \mathcal{H}^{n+1}_{R_*\otimes \mathrm{DL}/B_*}(\pi_*A;\pi_*\Omega^n F)$$

for $n \geq 1$, where $F = \mathrm{Fib}(B \to S)$. In particular,

(1) There are successively defined obstructions in $H^{n+1}_{R_*\otimes\mathrm{DL}/B_*}(\pi_*A;\pi_*\Omega^n F)$ for $n \geq 1$ to exhibiting a point of $\mathcal{C}\mathrm{Alg}_{R/S}^\phi(A,B)$;
(2) Once a point f of $\mathcal{C}\mathrm{Alg}_{R/S}^\phi(A,B)$ is chosen, there is a fringed spectral sequence of signature

$$E_1^{p,q} = H^{p-q}_{R_*\otimes\mathrm{DL}/B_*}(\pi_*A;\pi_*\Omega^p F) \Rightarrow \pi_q(\mathcal{C}\mathrm{Alg}_{R/S}(A,B),f), \qquad d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-1}.$$

PROOF. This is a direct application of Theorem 2.11 to $\mathcal{P} = \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}^{\mathrm{free}}$. $\square$

EXAMPLE 4.3. Let $\mathcal{F}\mathrm{in}_p$ denote the category of p-finite spaces, i.e. those spaces X such that X is truncated, $\pi_0 X$ is finite, and $\pi_n(X,x)$ is a finite p-group for all $x \in X$ and $n \geq 1$. By Mandell's p-adic homotopy theory [**Man01**], interpreted in the ∞-categorical context by Lurie [**Lur11a**], there is a fully faithful embedding

$$\mathcal{F}\mathrm{in}_p^{\mathrm{op}} \to \mathcal{C}\mathrm{Alg}_{\overline{\mathbb{F}}_p}, \qquad X \mapsto C(X;\overline{\mathbb{F}}_p),$$

where $C(X;\overline{\mathbb{F}}_p)$ is the spectrum of $\overline{\mathbb{F}}_p$-valued cochains on X, and this extends to a fully faithful embedding $\mathrm{Pro}(\mathcal{F}\mathrm{in}_p)^{\mathrm{op}} \simeq \mathrm{Ind}(\mathcal{F}\mathrm{in}_p^{\mathrm{op}}) \to \mathcal{C}\mathrm{Alg}_{\overline{\mathbb{F}}_p}$. In particular, given $X, Y \in \mathcal{G}\mathrm{pd}_\infty$ which are simply connected and of finite type, there is an equivalence

$$\mathrm{Map}(X, Y_p^\wedge) \simeq \mathcal{C}\mathrm{Alg}_{\overline{\mathbb{F}}_p}(C(Y;\overline{\mathbb{F}}_p), C(X,\overline{\mathbb{F}}_p)).$$

By Proposition 4.2, we may view the obstruction theory of Theorem 4.4 in this context as giving an unstable Adams spectral sequence. To note a special case, observe that the construction at the end of Subsection 4.1.4 easily translates to describe a free functor $S_\mathcal{U}\colon \mathrm{L}\mathcal{M}\mathrm{od}_\mathcal{U} \to \mathcal{R}\mathrm{ing}_\mathcal{U}^{\mathrm{aug}}$. In the Massey-Peterson case, where X and Y are pointed simply connected spaces of finite type and $H^*Y \cong S_\mathcal{U} M$ for some $M \in \mathrm{L}\mathcal{M}\mathrm{od}_\mathcal{U}$, this gives a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}_\mathcal{U}^{p-q}(M;\widetilde{H}^{*-p}X) \Rightarrow \pi_q \mathrm{Map}_*(X, Y_p^\wedge), \qquad d_r^{p,q}\colon E_r^{p,q}\colon E_r^{p+r,q-1}.$$

The Koszul complexes guaranteed by Proposition 4.3 recover the lambda complexes for computing these Ext groups. ◁

Example 4.4. Recall from Remark 3.3 that the forgetful functor $U\colon \mathrm{C}\mathcal{R}\mathrm{ing}_{\mathbb{F}_p} \to \mathcal{C}\mathrm{Alg}^{\mathrm{cn}}_{\mathbb{F}_p}$ is plethystic, and write its right adjoint as $\mathbb{A}^1$. Then for $R \in \mathcal{C}\mathrm{Alg}^{\mathrm{cn}}_{\mathbb{F}_p}$ there is an equivalence

$$\mathbb{A}^1(R) \simeq \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}(\mathbb{F}_p[t], R),$$

where $\mathbb{F}_p[t]$ has homotopy groups concentrated in degree 0. More generally, for any $a \in \mathbb{Z}$, we can consider the $\mathbb{F}_{p*}$-ring $S(e_a)$ as a differential graded algebra with trivial differential, in this way upgrade it to an object of $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$, and for $A \in \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$ consider the space $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}(S(e_a), A)$. The following are some comments about what Theorem 4.4 says about computing these spaces.

Observe first that the $\mathbb{F}_{p*}$-plethory DL is augmented over the initial $\mathbb{F}_{p*}$-plethory; this is not a purely formal fact, but is easily seen from the structure of DL. Restriction along the augmentation is itself a plethystic functor $\mathrm{C}\mathcal{R}\mathrm{ing}_{\mathbb{F}_{p*}} \to \mathcal{R}\mathrm{ing}_{\mathrm{DL}}$. More generally, there are plethystic functors $\mathrm{C}\mathcal{R}\mathrm{ing}_{R/S} \to \mathcal{R}\mathrm{ing}_{R\otimes \mathrm{DL}/S}$ for $R \in \mathcal{R}\mathrm{ing}^{\heartsuit}_{\mathrm{DL}}$ and $S \in \mathcal{R}\mathrm{ing}^{\heartsuit}_{R\otimes \mathrm{DL}}$; write G for the right adjoints to these. The filtration of $\mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}(S(e_a), R)$ guaranteed by Theorem 4.4 has layers that can now be identified as

$$\mathcal{H}^{n+1}_{\mathrm{DL}/R_*}(S(e_a), \pi_*\Omega^n R) \simeq \mathrm{Map}_{\mathrm{DL}/R_*}(S(e_a), B^{n+1}_{R_*}\pi_* R^{S^n_+}) \simeq G(B^{n+1}_{R_*}\pi_* R^{S^n_+})_a,$$

where $B^n_{R_*}$ denotes n-fold delooping in the slice category over R_*.

Consider for simplicity the case where R is augmented. Then the resulting spectral sequence for computing $\pi_*\mathcal{C}\mathrm{Alg}^{\mathrm{aug}}_{\mathbb{F}_p}(S(e_a), R)$ is of signature

$$E_1^{p,q} = \mathcal{H}^{p-q}_{\mathrm{DL}/R_*}(S(e_a); \pi_*\Omega^p R) \simeq \mathrm{Ext}^{p-q}_{\Delta}(e_a, s^{-p}R_*) \Rightarrow \pi_q\mathcal{C}\mathrm{Alg}^{\mathrm{aug}}_{\mathbb{F}_p}(S(e_a), R),$$

where Δ acts trivially on e_a.

For further simplicity, specialize to $p = 2$, and write $M = R_*$; we can describe the Koszul complex $K_\Delta(e_a, s^{-*}M)$ computing the above Ext groups as follows. Consider the space of tensors $\lambda_I \otimes m$ where $m \in M$ and $I = (r_1, \ldots, r_k)$ is a coadmissible sequence satisfying $r_1 < -a-1$ and $r_1 + \cdots + r_k + k \geq -m$. Now, complete this space to allow for infinite sums of the form $\sum_i \lambda_{I_i} \otimes m_i$ so long as for any fixed $n \in \mathbb{Z}$, there are finitely many nonzero terms involving m_i with $|m_i| = n$. This is $K_\Delta(e_a, s^{-*}M)$. The differential is given by

$$\delta(\lambda_I \otimes m) = \sum_{r\in\mathbb{Z}} \lambda_I\lambda_{-r-1} \otimes Q^r(m).$$

Return now to the special case of $\mathbb{A}^1$. Possibly more well-known is the subspace

$$\mathbb{G}_m(R) = \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}(\mathbb{F}_p[t^{\pm 1}], R) \simeq \mathbb{A}^1(R)^\times \subset \mathbb{A}^1(R)$$

of $\mathbb{A}^1(R)$ given by the strict units of R. All path components of $\mathbb{A}^1(R)$ and $\mathbb{G}_m(R)$ are equivalent, so these objects only differ in π_0. The Goerss-Hopkins spectral sequence computing $\pi_*\mathbb{G}_m(R)$, and relevant Koszul complex, has been studied by Fung [**Hou19**]. The perspective on $\mathbb{G}_m(R)$ afforded by viewing it as the spectrum of units of $\mathbb{A}^1(R)$ extends [**Hou19**, Proposition 3.11] to show that $\pi_n\mathbb{G}_m(R)$ is always an $\mathbb{F}_p$-vector space for $n > 0$. $\triangleleft$

4.1.7. André-Quillen-Goodwillie towers. Fix $R \in \mathcal{C}\mathrm{Alg}_{\mathbb{F}_p}$, and consider the category $\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}}$ of augmented $\mathbb{E}_\infty$ algebras over R. The functor $\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}} \to \mathcal{M}\mathrm{od}_R$ sending A to its augmentation ideal $\mathrm{Fib}(A \to R)$ is monadic; write $\widetilde{\mathbb{P}}$ for the associated monad on $\mathcal{M}\mathrm{od}_R$, given by $\widetilde{\mathbb{P}}M = \bigoplus_{n\geq 1}\mathbb{P}_n M$. For $A \in \mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}}$, one can consider in this context the topological André-Quillen homology and cohomology of A relative to R; write these as $\mathrm{TAQ}^R(A)$ and $\mathrm{TAQ}_R(A)$. Here, $\mathrm{TAQ}_R(A) = \mathcal{M}\mathrm{od}_R(\mathrm{TAQ}^R(A), R)$, and $\mathrm{TAQ}^R\colon \mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}} \to \mathcal{M}\mathrm{od}_R$ may be characterized as the unique functor which preserves geometric realizations and sends $\mathbb{P}M$ to M for $M \in \mathcal{M}\mathrm{od}_R$. Put another way, TAQ^R is left adjoint to restriction along the augmentation $\widetilde{\mathbb{P}} \to I$.

We can do something a little more general. Consider the augmentation ideal functor

$$U\colon \mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}} \to \mathcal{M}\mathrm{od}_R.$$

This has a Goodwillie tower, which we write as

$$U \to \cdots \to P_n^R \to P_{n-1}^R \to \cdots \to P_1^R \to 0.$$

The n'th term $P_n^R\colon \mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}} \to \mathcal{M}\mathrm{od}_R$ can be characterized as the unique functor which preserves geometric realizations and satisfies

$$P_n^R(\mathbb{P}M) = \widetilde{\mathbb{P}}M/\bigoplus_{k>n}\mathbb{P}_k M \simeq \bigoplus_{1\leq k\leq n}\mathbb{P}_k M.$$

In particular, $P_1^R = \mathrm{TAQ}^R$. See [**Kuh06**, Section 3] for more on this tower.

Following the recipe of Subsection 2.3.2, by restricting P_n^R to $\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug,free}}$ and taking homotopy groups we obtain a functor $\mathrm{h}\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug,free}} = \mathcal{R}\mathrm{ing}_{R_*\otimes\mathrm{DL}}^{\mathrm{aug,free}} \to \mathcal{M}\mathrm{od}_{R_*}^\heartsuit$. By left Kan extension, this extends to a functor

$$\overline{P}_n^{R_*}\colon \mathcal{R}\mathrm{ing}_{R_*\otimes\mathrm{DL}}^{\mathrm{aug},\heartsuit} \to \mathcal{M}\mathrm{od}_{R_*}^\heartsuit.$$

When $n = 1$, this is left adjoint to

$$\mathcal{M}\mathrm{od}_{R_*} \to \mathcal{R}\mathrm{ing}_{R_*\otimes\mathrm{DL}}^{\mathrm{aug}}, \qquad M_* \mapsto R_* \ltimes \overline{M}_*.$$

Thus, where $\epsilon\colon R_*\otimes\Delta \to R_*$ is the augmentation and $Q_{R_*}\colon \mathcal{R}\mathrm{ing}_{R_*\otimes\mathrm{DL}}^{\mathrm{aug}} \to \mathrm{L}\mathcal{M}\mathrm{od}_{R_*\otimes\Delta}$ is the functor of indecomposables, there is an equivalence

$$\mathbb{L}\overline{P}_1^{R_*} \simeq \epsilon_!\mathbb{L}Q_{R_*},$$

where $\epsilon_!$ is considered in the derived sense. This can be computed via a bar construction, as in Subsection 3.2.3.

THEOREM 4.5. Fix notation as above, and fix $A \in \mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}}$.

(1) There is a convergent spectral sequence in $\mathcal{M}\mathrm{od}_{R_*}^{\heartsuit}$ of signature

$$E^1_{p,q} = s^q \mathbb{L}_{p+q}\overline{P}^{R_*}_n(A_*) \Rightarrow \pi_{*+p}P^R_n(A), \qquad d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q-1};$$

(2) There is a conditionally convergent spectral sequence of signature

$$E_1^{p,q} = \mathrm{Ext}^{p+q}_{R_*\otimes\Delta}(\mathbb{L}Q_{R_*}(A_*), \overline{R}_{*+p}) \Rightarrow \mathrm{TAQ}^q_R(A), \qquad d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q+1}.$$

PROOF. Given the preceding discussion, the first spectral sequence is obtained by an application of Theorem 2.7. The second spectral sequence can be obtained by patching together the filtrations of $\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}}(A, R \ltimes \Sigma^n\overline{R})$ for various n given by Theorem 4.4 and applying the isomorphisms $H^{p+q}_{R_*\otimes\mathrm{DL}/R_*}(A_*; \pi_*\Omega^p\overline{R}) \cong \mathrm{Ext}^{p+q}_{R_*\otimes\Delta}(\mathbb{L}Q_{R_*}(A_*), \overline{R}_{*+p})$. □

A form of this spectral sequence was originally constructed by Basterra [**Bas99**].

EXAMPLE 4.5. Work of Miller [**Mil78**] produces a spectral sequence converging to H_*X for a connective spectrum X, with initial page depending on $H_*\Omega^\infty X$ as a ring over the Dyer-Lashof algebra. This can be understood from the perspective of Theorem 4.5: there is an equivalence

$$\mathrm{TAQ}^{\mathbb{F}_p}(\mathbb{F}_p \otimes \Sigma^\infty_+\Omega^\infty X) \simeq \mathbb{F}_p \otimes X$$

when X is connective [**Kuh06**, Example 3.9], and thus Theorem 4.5 gives a spectral sequence

$$E^1_{p,q} = s^q\mathbb{L}_{p+q}\overline{P}^{\mathbb{F}_{p*}}_1(H_*\Omega^\infty X) = s^q\pi_{p+q}\epsilon_!\mathbb{L}Q(H_*\Omega^\infty X) \Rightarrow H_{*+p}X.$$

The underlying ring of $H_*\Omega^\infty X$ splits as

$$H_*\Omega^\infty X = \mathbb{F}_p[\pi_0 X] \otimes H_*\Omega^\infty_0 X,$$

where $\mathbb{F}_p[\pi_0 X]$ is an abelian group ring and $H_*\Omega^\infty_0 X$ is a connected Hopf algebra. The structure theory of graded bicommutative Hopf algebras implies that $\mathbb{L}Q(H_*\Omega^\infty X)$ is always 1-truncated, and so the E^1-page of this spectral sequence is somewhat accessible even when $H_*\Omega^\infty X$ is not smooth. ◁

4.2. Lubin-Tate spectra

In this section we apply the machinery developed in the preceding chapters to the study of $K(h)$-local $\mathbb{E}_\infty$ algebras over a Lubin-Tate spectrum.

There are some additional preliminaries to cover before these applications. In Subsection 4.2.1, we give the notion of an even-periodic plethory, and in Subsection 4.2.2, we explain how a cobialgebroid over an ordinary ring, under some niceness assumptions, gives rise to a formal category scheme. These topics are present in some form in Rezk's work [**Rez09**], which also describes the general structure of power operations for Lubin-Tate spectra, and in Subsection 4.2.3 we recall the relevant parts of this story. In Subsection 4.2.4, we explain how to deal with the completions that arise from the $K(h)$-local condition.

With all this in place, we reap the benefits in Subsection 4.2.5 and Subsection 4.2.6. In Subsection 4.2.5, we describe what the mapping space obstruction theory of Theorem 2.11 looks like for $K(h)$-local $\mathbb{E}_\infty$ algebras over Lubin-Tate spectra. This obstruction theory turns out to be very pleasant at heights $h \leq 2$, with frequently vanishing obstruction groups, and we apply this to produce new $\mathbb{E}_\infty$ complex orientations. In Subsection 4.2.6, we describe an analogue of the Basterra spectral sequence for computing topological André-Quillen homology and cohomology in this context.

4.2.1. Even-periodic plethories. Fix an ordinary $\mathbb{Z}$-graded commutative ring $R = R_*$, with associated category $\mathcal{M}\mathrm{od}_R = \mathcal{M}\mathrm{od}_{R_*}$ of $\mathbb{Z}$-graded modules. Consider $\mathcal{M}\mathrm{od}_R$ as a symmetric monoidal category with symmetrizer employing the Koszul sign rule, and abbreviate both $\otimes_{R_*}$ and $\otimes_{R_0}$ as just $\otimes$. Write E for a copy of R generated in degree 1.

Definition 4.5.

(1) R is *even-periodic* if $R_1 = 0$, and for all $k \in \mathbb{Z}$, the map $R_k \otimes R_2 \to R_{k+2}$ is an isomorphism. The following definitions will be made under the assumption that R is even-periodic.

(2) An R-cobialgebroid Γ is *even-periodic* if
   (a) As a functor, Γ preserves the full subcategory of $\mathcal{M}\mathrm{od}_R^\heartsuit$ spanned by those R-modules which are concentrated in even degrees;
   (b) We have chosen a map $\Gamma^E \to \Gamma$ equipping Γ with suspensions (Definition 3.11) which is an isomorphism when restricted to the full subcategory $\mathcal{M}\mathrm{od}_R^\heartsuit$ spanned by those R-modules which are concentrated in even degrees.

(3) An R-plethory Λ is *even-periodic* if
   (a) The underlying exponential monad of Λ preserves the full subcategory of $\mathcal{M}\mathrm{od}_R^\heartsuit$ spanned by those R-modules which are concentrated in even degrees;
   (b) We have chosen an isomorphism $\Delta(\Lambda)^E \to \Gamma(\Lambda)$ which equips Λ with suspensions (Definition 3.11) and makes $\Gamma(\Lambda)$ into an even-periodic cobialgebroid. $\triangleleft$

For the rest of this subsection, R is assumed to be even-periodic. Because R is even-periodic, multiplication gives an isomorphism $R_{-2} \otimes R_2 \to R_0$, and so R_2 is an invertible R_0-module. Denote this module by L. The ring R may then be identified as $R = \bigoplus_{n\in\mathbb{Z}} L^n$, where $L^n = L^{\otimes n}$ consists of elements in degree $2n$.

Let $\mathcal{M}\mathrm{od}_{R_\star} = \mathcal{M}\mathrm{od}_{R_0}^{\times 2}$ denote the category of $\mathbb{Z}/(2)$-graded R_0-modules. Then the functor

$$\mathcal{M}\mathrm{od}_R \to \mathcal{M}\mathrm{od}_{R_\star}, \qquad M_* \mapsto (M_0, M_{-1})$$

is an equivalence of categories, for it has essential inverse

$$(M_0, M_{-1}) \mapsto M_*, \qquad M_{2n-\epsilon} = L^n \otimes M_{-\epsilon}.$$

In particular, $\mathcal{M}\mathrm{od}_{R_0}$ may be identified with the full subcategory of $\mathcal{M}\mathrm{od}_R$ spanned by those R-modules which are concentrated in even degrees. Under the above equivalence, the symmetric monoidal structure on $\mathcal{M}\mathrm{od}_R$ transfers to a symmetric monoidal structure on $\mathcal{M}\mathrm{od}_{R_\star}$ whose tensor product may be identified as

$$\begin{aligned}(M_0, M_{-1}) \otimes (M_0', M_{-1}') = (&M_0 \otimes M_0' \oplus L \otimes M_{-1} \otimes M_{-1}', \\ &M_0 \otimes M_{-1}' \oplus M_{-1} \otimes M_0'),\end{aligned}$$

where the symmetrizer acts with a sign on L.

Let $L^{1/2} = E^{-1}$, considered as either an object of $\mathcal{M}\mathrm{od}_R$ or $\mathcal{M}\mathrm{od}_{R_\star}$. Then

$$L^{1/2} = (0, R_0), \qquad L^{1/2} \otimes L^{1/2} = L^1 = (L, 0).$$

So for every $M \in \mathcal{M}\mathrm{od}_R$ there are unique R_0-modules M_0 and M_{-1} such that

$$M \cong M_0 \oplus L^{1/2} \otimes M_{-1}.$$

Fix next an even-periodic cobialgebroid Γ, and abbreviate $\Gamma_{n,m} = \Gamma(E^n R)(E^m R)$. Even-periodicity of R implies

$$L \otimes_l \Gamma_{n,m} \cong \Gamma_{n,m+2}, \qquad \Gamma_{n,m}\, {}_r\!\otimes L \cong \Gamma_{n-2,m}.$$

Here, each $\Gamma_{n,m}$ is an R_0-bimodule, and we have used subscripts to indicate which R_0-module structure we are taking a tensor product with respect to. The assumption that Γ preserves even objects implies that $\Gamma_{n,m} = 0$ unless $n \equiv m \pmod 2$. The suspension maps for Γ are maps $\Gamma_{n-1,m-1} \to \Gamma_{n,m}$; even-periodicity of Γ implies that these are isomorphisms when n and m are even, and they are algebra maps when $n = m$. The algebra structure on Γ is thus essentially determined by $\Gamma_{0,0}$, and there is an equivalence of categories $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star}$ overlying the equivalence $\mathcal{M}\mathrm{od}_R \simeq \mathcal{M}\mathrm{od}_{R_\star}$, where $\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star} = \mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_{0,0}}^{\times 2}$ is the category of $\mathbb{Z}/(2)$-graded $\Gamma_{0,0}$-modules.

The coproduct on Γ is encoded by maps

$$\Delta^\times \colon \Gamma_{n+n',m+m'} \to \Gamma_{n,m}\, {}_l\!\otimes_l \Gamma_{n',m'},$$

which for $n = n' = 0 = m = m'$ contribute to the R_0-cobialgebroid structure of $\Gamma_{0,0}$. As $\mathcal{M}\mathrm{od}_{\Gamma_\star}$ is strongly monoidal over $\mathcal{M}\mathrm{od}_{R_\star}$, its tensor product must take the form

$$\begin{aligned}(M_0, M_{-1}) \otimes (M_0', M_{-1}') = (&M_0 \otimes M_0' \oplus L \otimes M_{-1} \otimes M_{-1}', \\ &M_0 \otimes M_{-1}' \oplus M_{-1} \otimes M_0').\end{aligned}$$

But this does not fully describe the tensor product: missing is a description of the $\Gamma_{0,0}$-module structure. On the summands $M_0' \otimes M_0''$, $M_0' \otimes M_{-1}''$, and $M_{-1}' \otimes M_0''$, this action is obtained just from the coproduct on $\Gamma_{0,0}$ and isomorphism $\Gamma_{-1,-1} \cong \Gamma_{0,0}$, so consider the remaining summand $L \otimes M_{-1}' \otimes M_{-1}''$. Here by definition the action arises via the map

$$\Gamma_{0,0} \cong L \otimes_l \Gamma_{-2,-2}\, {}_r\!\otimes L^{-1} \to L \otimes_l \Gamma_{-1,-1}\, {}_l\!\otimes_l \Gamma_{-1,-1}\, {}_r\!\otimes L^{-1} \cong L \otimes_l \Gamma_{0,0}\, {}_l\!\otimes_l \Gamma_{0,0}\, {}_r\!\otimes L^{-1}.$$

On the other hand, there is an action of $\Gamma_{0,0}$ on $L \otimes M_{-1} \otimes M_{-1}'$ obtained from the iterated coproduct of $\Gamma_{0,0}$ and the action of $\Gamma_{0,0}$ on L by way of the double suspension $\Gamma_{0,0} \to \Gamma_{2,2}$. These actions agree; the definition of a cobialgebroid with suspensions was chosen in order to make this so. This gives a full understanding of $\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star}$ as a symmetric monoidal category; we can summarize the situation as follows.

Proposition 4.4. Let Γ be an even-periodic R-cobialgebroid. In particular, $\Gamma_{0,0}$ is an R_0-cobialgebroid, and there is a chosen $\Gamma_{0,0}$-module structure on $L = R_2$. Then there is an equivalence of symmetric monoidal categories

$$\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star}, \qquad M_* \mapsto (M_0, M_{-1}),$$

where $\mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_\star} = \mathrm{L}\mathcal{M}\mathrm{od}_{\Gamma_{0,0}}^{\times 2}$ is the category of $\mathbb{Z}/(2)$-graded $\Gamma_{0,0}$-modules, with symmetric monoidal product given by

$$\begin{aligned}(M_0, M_{-1}) \otimes (M_0', M_{-1}') = (&M_0 \otimes M_0' \oplus L \otimes M_{-1} \otimes M_{-1}', \\ &M_0 \otimes M_{-1}' \oplus M_{-1} \otimes M_0'),\end{aligned}$$

where $\Gamma_{0,0}$ acts on each term through its coproduct and the symmetrizer acts on L with a sign. $\square$

Remark 4.4. Note that although L is an invertible R_0-module, it is generally not invertible as a $\Gamma_{0,0}$-module. $\triangleleft$

Remark 4.5. Let A be an object of $\mathcal{R}\mathrm{ing}_{\Gamma_\star}^{\heartsuit}$. Given $x, y \in A_{-1}$, one may wish to form their product xy, and to consider how the elements of $\Gamma_{0,0}$ act on this product. However $xy \in A_{-2}$, i.e. this product takes us outside the $\mathbb{Z}/(2)$-graded setting. To remain in the $\mathbb{Z}/(2)$-graded setting, it is more correct to say that the product of two elements of A_{-1} is given by a map

$$L \otimes A_{-1} \otimes A_{-1} \to A_0$$

of $\Gamma_{0,0}$-modules. If L is trivializable, then by choosing a trivialization we may treat this as a map $A_{-1} \otimes A_{-1} \to A_0$, but one must not forget the presence of L in considering how this map interacts with Γ. $\triangleleft$

Example 4.6. Let Γ be the $R = W(\kappa)[[a]]$-cobialgebroid of Example 3.14 and Example 3.18. Then Γ upgrades to an even-periodic cobialgebroid over the even-periodic ring $R[u^{\pm 1}]$, where $|u| = 2$. The Γ-module structure on $\omega = L = R\{u\}$ encoding this is given by

$$Q_0 u = 0, \qquad Q_1 u = -u, \qquad Q_2 u = 0.$$

If M is a Γ-module, then $\omega \otimes M$ consists of elements um for $m \in M$, and has Γ-module structure

$$Q_0(um) = -2uQ_2(m), \quad Q_1(um) = -uQ_0(m) - auQ_2(m), \quad Q_2(um) = -uQ_1(m).$$

A $\mathbb{Z}/(2)$-graded Γ-ring is then a $\mathbb{Z}/(2)$-graded R-ring A equipped with an action of Γ such that if either $x \in A_0$ or $y \in A_0$, then Γ acts on xy via Example 3.18; and if $x, y \in A_{-1}$, then Γ acts on $xy \in A_0$ by

$$\begin{aligned}
Q_0(xy) =& -2Q_0(x)Q_2(y) - 2Q_2(x)Q_0(y) - 2Q_1(x)Q_1(y) - 2aQ_2(x)Q_2(y), \\
Q_1(xy) =& -Q_0(x)Q_0(y) - 2Q_1(x)Q_2(y) - 2Q_2(x)Q_1(y) \\
& - aQ_0(x)Q_2(y) - aQ_2(x)Q_0(y) - aQ_1(x)Q_1(y) - a^2Q_2(x)Q_2(y), \\
Q_2(xy) =& -Q_0(x)Q_1(y) - Q_1(x)Q_0(y) \\
& - aQ_1(x)Q_2(y) - aQ_2(x)Q_1(y) - 2Q_2(x)Q_2(y).
\end{aligned}$$

See especially Example 4.7 and Example 4.10 for more on this example. $\triangleleft$

Now fix an even-periodic R-plethory Λ, and abbreviate $\Gamma = \Gamma(\Lambda)$ and $\Delta = \Delta(\Lambda)$. We can picture the relevant suspension maps as fitting into a diagram

$$\begin{tikzcd}
 & \Gamma_{-1,-1} \arrow[d, "\simeq"] & \Gamma_{0,0} \arrow[d] & \Gamma_{1,1} \arrow[d, "\simeq"] & \Gamma_{2,2} \arrow[d] & \Gamma_{3,3} \\
\Delta_{-2,-2} \arrow[ur, "\simeq"] & \Delta_{-1,-1} \arrow[ur, "\simeq"] & \Delta_{0,0} \arrow[ur, "\simeq"] & \Delta_{1,1} \arrow[ur, "\simeq"] & \Delta_{2,2} \arrow[ur, "\simeq"] &
\end{tikzcd}.$$

As R is even-periodic, there are canonical equivalences $\mathrm{LMod}_{\Gamma_{n,n}} \simeq \mathrm{LMod}_{\Gamma_{n+2,n+2}}$ and $\mathrm{LMod}_{\Delta_{n,n}} \simeq \mathrm{LMod}_{\Delta_{n+2,n+2}}$ for each $n \in \mathbb{Z}$, given by by tensoring with L. Coupling these with the isomorphisms in the above diagram yields the following.

Proposition 4.5. There is a canonical Morita equivalence $\mathrm{LMod}_\Delta \simeq \mathrm{LMod}_\Gamma$, and the composite $\mathrm{LMod}_\Gamma \simeq \mathrm{LMod}_\Delta \to \mathrm{LMod}_\Gamma$, the second functor being restriction along $\Gamma \to \Delta$, is given by $L^{1/2} \otimes -$. $\square$

Although LMod_Δ is what appears when considering the Quillen cohomology of Λ-rings, in many examples of interest it is possible to produce a partial section of $L^{1/2}\otimes -\colon \mathrm{LMod}_\Gamma \to \mathrm{LMod}_\Gamma$, thereby allowing one to reduce to computations in LMod_Γ. See Example 4.10 for an example.

4.2.2. Quasicoherent sheaves. Fix an ordinary commutative ring R, abbreviating $\otimes = \otimes_R$, and fix an R-cobialgebroid Γ. If we forget the algebra and right R-module structures on Γ, then we are left with nothing more than a (counital, coassociative, cocommutative) R-coalgebra. Under suitable niceness assumptions, R-coalgebras give one approach to the theory of formal schemes over R; this is best known when R is a field and these niceness assumptions are automatic, see for instance [**Dem72**, Section I.6]. It turns out that the entire R-cobialgebroid structure of Γ may be understood this way, at least under suitable niceness assumptions.

In the following, we will freely use the language of formal schemes as developed in [**Str99**], particularly the technical notions of solid formal schemes and coalgebraic formal schemes; however, we will write $\mathrm{Sp}^\vee$ for what is written there as sch, informally defined as $\mathrm{Sp}^\vee C = \mathrm{Spf}(C^\vee)$ for an R-coalgebra C with good basis. We abbreviate "coalgebra with good basis" to "good coalgebra", and write $\widehat{\otimes}$ for the completed tensor product of pro-R-modules.

Proposition 4.6. Fix an R-cobialgebroid Γ which is good as an R-coalgebra. Then the pair $(\mathrm{Spec}\, R, \mathrm{Sp}^\vee \Gamma)$ naturally carries the structure of a formal category scheme.

Proof. We must describe the structure of a category object on the pair $(\mathrm{Spec}\, R, \mathrm{Sp}^\vee \Gamma)$. The source map $s\colon \mathrm{Sp}^\vee \Gamma \to \mathrm{Spec}\, R$ is simply the map arising from the definition of $\mathrm{Sp}^\vee \Gamma$ as a formal R-scheme. In describing the remaining maps, we will make use of the fact that $\mathrm{Sp}^\vee \Gamma$ is a solid formal scheme, so that it suffices to work with $\Gamma^\vee = \mathrm{LMod}_R(\Gamma, R)$ as a formal ring. The target map $t\colon \mathrm{Sp}^\vee \Gamma \to \mathrm{Spec}\, R$ is dual to the map of formal rings

$$t\colon R \to \Gamma^\vee, \qquad t(r)(\gamma) = \epsilon(\gamma r).$$

The unit map $\iota\colon \mathrm{Spec}\, R \to \mathrm{Sp}^\vee \Gamma$ is dual to the map of formal rings

$$\iota\colon \Gamma^\vee \to R, \qquad \iota(f) = f(1).$$

To define $c\colon \mathrm{Sp}^\vee \Gamma_s \times_{\mathrm{Spec}\, R, t} \mathrm{Sp}^\vee \Gamma \to \Gamma$, observe first that $\mathrm{Sp}^\vee \Gamma\,{}_s{\times}_{\mathrm{Spec}\, R,t} \mathrm{Sp}^\vee \Gamma$ is a solid formal scheme represented by $\Gamma^\vee\,{}_s\widehat{\otimes}_t\,\Gamma^\vee$. As Γ admits a good basis, it may be written as $\Gamma \cong \mathrm{colim}_\alpha \Gamma_\alpha$ where $\Gamma_\alpha \subset \Gamma$ is a standard coalgebra, in which case $\Gamma^\vee \cong \lim_\alpha \Gamma_\alpha^\vee$ as a formal ring with each $\Gamma_\alpha^\vee$ discrete and finitely generated free as a right R-module. It follows that

$$\begin{aligned}
\Gamma^\vee{}_s\widehat{\otimes}_t\Gamma^\vee &\cong (\lim_\alpha \Gamma_\alpha^\vee){}_s\widehat{\otimes}_t\Gamma^\vee \\
&\cong \lim_\alpha(\Gamma_\alpha^\vee{}_s\otimes_t\Gamma^\vee) \\
&\cong \lim_\alpha(\Gamma{}_r\otimes_l\Gamma_\alpha)^\vee \cong (\Gamma{}_r\otimes_l\Gamma)^\vee.
\end{aligned}$$

A similar argument can be used to show that $\Gamma{}_r\otimes_l\Gamma$ is itself a good R-coalgebra, and the above then gives an isomorphism

$$\mathrm{Sp}^\vee\Gamma{}_s\times_{\mathrm{Spec}\,R,t}\mathrm{Sp}^\vee\Gamma \cong \mathrm{Sp}^\vee(\Gamma{}_r\otimes_l\Gamma)$$

of formal R-schemes. The composition map is now dual to the product on Γ.

That $(\mathrm{Spec}\,R, \mathrm{Sp}^\vee\Gamma)$ is a category scheme with this structure amounts to a direct translation between definitions. □

Remark 4.6. Fix a good R-cobialgebroid Γ. As algebras for the monad Γ are equivalent to coalgebras for the comonad $\Gamma^\vee$, we may encode a Γ-module as an R-module equipped with a coaction $P\colon M \to \Gamma^\vee(M) \cong \Gamma^\vee{}_s\widehat{\otimes}\, M$ satisfying the evident counity and coassociativity conditions. The coaction P is left R-linear, where the left R-module structure on $\Gamma^\vee$ is through the target map $t\colon R \to \Gamma^\vee$.

By definition, an object of $\mathcal{R}\mathrm{ing}_\Gamma^\heartsuit$ is an R-ring A equipped with a Γ-module structure satisfying the Cartan formulas encoded by the coproduct $\Gamma \to \Gamma{}_l\otimes_l\Gamma$. When the Γ-module structure on A is encoded by a coaction $P\colon A \to \Gamma^\vee{}_s\widehat{\otimes}\, A$, this is equivalent to the condition that P is a homomorphism of rings. ◁

Example 4.7. Let Γ be the R-cobialgebroid of Example 3.14 and Example 3.18, and suppose for simplicity that $\kappa = \mathbb{F}_2$ so that $R = \mathbb{Z}_2[[a]]$. The length grading $\Gamma = \bigoplus_{n\geq 0}\Gamma[n]$ is a decomposition of R-coalgebras, so as each $\Gamma[n]$ is finitely generated and free over R, the coalgebra Γ is good, and

$$\mathrm{Sp}^\vee\Gamma \cong \coprod_{n\geq 0}\mathrm{Spec}\,\Gamma[n]^\vee.$$

As Γ is quadratic, a Γ-module is determined by an R-module M equipped with a left R-linear map $P\colon M \to \Gamma[1]^\vee{}_s\otimes M$ such that there exists a factorization through the dashed arrow in the diagram

$$\begin{tikzcd}
M \arrow[r, "P"] \arrow[d, dashed] & \Gamma[1]^\vee{}_s\otimes M \arrow[d, "\Gamma[1]^\vee\otimes P"] \\
\Gamma[2]^\vee{}_s\otimes M \arrow[r, "c\otimes M"] & \Gamma[1]^\vee{}_s\otimes_t\Gamma[1]^\vee{}_s\otimes M
\end{tikzcd}.$$

There is an isomorphism of rings

$$\Gamma[1]^\vee \cong R[d]/(d^3 = ad + 2),$$

where the basis $\Gamma[1]^\vee = \{1, d, d^2\}R$ is dual to the basis $\Gamma[1] = R\{Q_0, Q_1, Q_2\}$. The map t is the ring homomorphism

$$t\colon R \to \Gamma[1]^\vee, \qquad t(a) = a^2 + 3d - ad^2,$$

and R-linearity of P is with respect to t, i.e. $P(am) = P(m)(a^2 + 3d - ad^2)$ for $m \in M$.

The category $\mathcal{R}\mathrm{ing}_\Gamma^\heartsuit$ is more pleasantly understood from this perspective. If A is an R-ring and Γ-module, with structure map $P\colon A \to \Gamma[1]^\vee{}_s\!\otimes A \cong A[d]/(d^3 = ad + 2)$, then A is a Γ-ring precisely when P is a ring homomorphism.

It is possible to compute $\Gamma[2]^\vee$ directly given our knowledge of Γ. Doing so is backwards, as Γ is computed in [**Rez08**] by computing in the formal category scheme $(\operatorname{Spf} R, \operatorname{Sp}^\vee \Gamma)$, whose interpretation we recall in Subsection 4.2.3. Better yet, as explained in [**Rez13**], one may avoid directly dealing with $\Gamma[2]^\vee$ altogether. Write

$$\Gamma[1]^\vee{}_s\!\otimes_t \Gamma[1]^\vee \cong R[d', d]/(d^3 = ad + 2, d'^3 = (a^2 + 3d - ad^2)d' + 2).$$

Then there is a Cartesian square

$$\begin{tikzcd}
\Gamma[2]^\vee \arrow[r, "c"] \arrow[d, "\epsilon"] & \Gamma[1]^\vee{}_s\!\otimes_t \Gamma[1]^\vee \arrow[d, "f"] \\
R \arrow[r, "s"] & \Gamma[1]^\vee
\end{tikzcd}$$

of rings, where f is the right R-linear map given by $f(d) = d$ and $f(d') = a - d^2$. Thus an R-linear map $P\colon M \to \Gamma[1]^\vee{}_s\!\otimes M$ makes M into a Γ-module precisely when there exists a (necessarily unique) map Ψ filling in

$$\begin{tikzcd}
M \arrow[r, "P"] \arrow[dd, dashed, "\Psi"] & \Gamma[1]^\vee{}_s\!\otimes M \arrow[d, "\Gamma[1]^\vee\otimes P"] \\
& \Gamma[1]^\vee{}_s\!\otimes_t \Gamma[1]^\vee{}_s\!\otimes M \arrow[d, "f\otimes M"] \\
M \arrow[r, "s\otimes M"] & \Gamma[1]^\vee{}_s\!\otimes M
\end{tikzcd} .$$

When $M = R$, the map Ψ happens to be the identity. This implies that Ψ is R-linear in general, although it need not be the identity in general.

See in particular Example 4.10 for more on this example. $\triangleleft$

Let $\mathcal{M}\mathrm{od}^\heartsuit\colon \mathrm{C}\mathcal{R}\mathrm{ing}^\heartsuit \to \mathcal{C}\mathrm{at}$ denote the pseudofunctor

$$R \mapsto \mathcal{M}\mathrm{od}_R^\heartsuit, \qquad (f\colon R \to S) \mapsto (S \otimes_R -\colon \mathcal{M}\mathrm{od}_R^\heartsuit \to \mathcal{M}\mathrm{od}_S^\heartsuit).$$

Given some other pseudofunctor $\mathfrak{X}\colon \mathrm{C}\mathcal{R}\mathrm{ing}^\heartsuit \to \mathcal{C}\mathrm{at}$, define $\mathcal{Q}\mathcal{C}\mathrm{oh}(\mathfrak{X})^\heartsuit$ to be the category of pseudonatural transformations $\mathfrak{X}^{\mathrm{op}} \to \mathcal{M}\mathrm{od}^\heartsuit$. This is an additive symmetric monoidal category.

Proposition 4.7. Let Γ be a good R-cobialgebroid, and $\mathcal{X} = (\operatorname{Spec} R, \operatorname{Sp}^\vee \Gamma)$. Then there is an equivalence $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit \simeq \mathcal{Q}\mathcal{C}\mathrm{oh}(\mathcal{X})^\heartsuit$ of symmetric monoidal categories.

Proof. There is a well-known symmetric monoidal equivalence between the category of comodules for a commutative Hopf algebroid and the category of quasicoherent sheaves on the associated presheaf of groupoids [**Hov02**]. The claim at hand is no different from this, so we will indicate the construction but omit detailed verifications of naturality.

The equivalence $\mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit \to \mathcal{Q}\mathcal{C}\mathrm{oh}(\mathcal{X})^\heartsuit$ is constructed as follows. Fix $M \in \mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit$, so we wish to construct a pseudonatural transformation $\mathcal{F}_M\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{M}\mathrm{od}^\heartsuit$. Fix a ring S; then the functor $\mathcal{F}_M^S\colon \mathcal{X}(S)^{\mathrm{op}} \to \mathcal{M}\mathrm{od}_S^\heartsuit$ is defined as follows. Fix $g \in \mathcal{X}(S)$ realized by a map $g\colon R \to S$. Then $\mathcal{F}_M^S$ is given on objects by $\mathcal{F}_M^S(g) = S\,{}_g\!\otimes M$. Fix $\alpha\colon g' \to g$ in $\mathcal{X}(S)$, realized by a diagram

$$\begin{tikzcd} & \operatorname{Spec} R \\ \operatorname{Spec} S \arrow[ur, "g"] \arrow[r, "\alpha"] \arrow[dr, "g'"'] & \operatorname{Sp}^\vee \Gamma \arrow[u, "t"'] \arrow[d, "s"] \\ & \operatorname{Spec} R \end{tikzcd} .$$

Then α is dual to a map $\alpha\colon \Gamma^\vee \to S$ of formal rings, i.e. one that factors through some discrete quotient $\Gamma_\alpha^\vee$, where $\Gamma_\alpha \subset \Gamma$ is a standard coalgebra. Then $\mathcal{F}_M^S$ is given on morphisms by declaring $\mathcal{F}_M^S(g)$ to be the composite

$$\begin{aligned}\mathcal{F}_M(g) = S\,{}_g\!\otimes M \to S\,{}_g\!\otimes \Gamma^\vee(M) \cong S\,{}_g\!\otimes_t \Gamma^\vee\,{}_s\widehat{\otimes}\, M \\ \to S\,{}_g\!\otimes_g S\,{}_{g'}\!\otimes M \to S\,{}_{g'}\!\otimes M = \mathcal{F}_M(g').\end{aligned}$$

The inverse equivalence $\mathcal{Q}\mathcal{C}\mathrm{oh}(\mathcal{X})^\heartsuit \to \mathrm{L}\mathcal{M}\mathrm{od}_\Gamma^\heartsuit$ is constructed as follows. Fix $\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{M}\mathrm{od}^\heartsuit$. Let $i \in \mathcal{X}(R)$ be classified by the identity of R, and write $M = \mathcal{F}_R(i)$. A $\Gamma^\vee$-comodule structure on M can be defined as follows. Note first that $\mathcal{X}$ extends to a functor on pro-rings in the evident way; in particular $\mathcal{X}(\Gamma^\vee)$ is a category, and there are elements $s, t \in \mathcal{X}(\Gamma^\vee)$ classified by the source and target maps of $\Gamma^\vee$. The identity map of $\Gamma^\vee$ corresponds to a map $c\colon s \to t$ in $\mathcal{X}(\Gamma^\vee)$, and this gives a $\Gamma^\vee$-linear map

$$\Gamma^\vee\,{}_t\widehat{\otimes}\, M \to \Gamma^\vee\,{}_s\widehat{\otimes}\, M \cong \Gamma^\vee(M).$$

This is adjoint to a map $M \to \Gamma^\vee(M)$ which defines a Γ-module structure on M, and the inverse equivalence sends $\mathcal{F}$ to M with this Γ-module structure. $\square$

Example 4.8. Let $\sigma\colon R \to R$ be a ring homomorphism, and consider the R-cobialgebroid

$$\Gamma = R\langle\psi\rangle/(\psi \cdot r = \sigma(r)\cdot\psi), \qquad \Delta^\times(\psi) = \psi\otimes\psi, \qquad \epsilon(\psi) = 1;$$

compare Example 3.19. Then

$$\Gamma \cong \bigoplus_{n\geq 0} R\{\psi^n\}$$

with ψ^n grouplike, so

$$\mathrm{Sp}^\vee\,\Gamma \cong \coprod_{n\geq 0} \operatorname{Spec} R.$$

The target map on the n'th component is given by restriction along σ^n. The formal category scheme $\mathcal{X}\colon \mathcal{CR}\mathrm{ing}^\heartsuit \to \mathcal{C}\mathrm{at}$ obtained from this R-cobialgebroid sends a ring S to the category $\mathcal{X}(S)$ identified as follows. An object of $\mathcal{X}(S)$ is a map $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$. Given $f, f' \in \mathcal{X}(S)$, a morphism $\alpha\colon f \to f'$ is a decomposition $\operatorname{Spec} S = \coprod_{0\leq n \ll \infty} \operatorname{Spec}(S_n)$ such that $f'|_{\operatorname{Spec}(S_n)} = (\sigma^n)^* f$. A quasicoherent sheaf on $\mathcal{X}$ is an R-module equipped with a σ-semilinear homomorphism $\psi\colon M \to M$, i.e. an additive map such that $\psi(r\cdot m) = \sigma(r)\cdot\psi(m)$ for $r \in R$ and $m \in M$. This is a quasicoherent sheaf of rings if M is a ring and ψ is a ring homomorphism. $\triangleleft$

4.2.3. Power operations for Morava E-theory. Let κ be a perfect field of positive characteristic p, and $\mathbb{G}_0 \to \operatorname{Spec}(\kappa) = X_0$ be a formal group of finite height h. Let $\mathbb{G} \to X$ be the universal Lubin-Tate deformation [**LT66**] of this formal group, and E be the associated Lubin-Tate spectrum, also referred to as a spectrum of Morava E-theory. Write $\mathfrak{m} \subset E_0 = \mathcal{O}_X$ for the maximal ideal.

By the Goerss-Hopkins-Miller theorem [**GH04**] [**GH05**], E is a $K(h)$-local even-periodic $\mathbb{E}_\infty$ ring spectrum, and this construction is functorial in the input $(\kappa, \mathbb{G}_0)$ and fiberwise isomorphisms. There results a theory of E-power operations acting on the homotopy groups of $K(h)$-local $\mathbb{E}_\infty$ algebras over E, and these are now well-understood conceptually owing to work of Ando, Hopkins, Strickland, and Rezk. The formulation by Rezk [**Rez09**], building on work of Strickland [**Str98**], itself building on calculations of Kashiwabara [**Kas98**], is the most convenient approach for our purposes. It seems easiest, both for the writer and the reader, to collect what we need in one place, so we will summarize some of the structure of these operations in one big statement.

Write $\widehat{\mathbb{P}}$ for the free $K(h)$-local $\mathbb{E}_\infty$ algebra monad on $\mathcal{M}\mathrm{od}_E$, so that there is a decomposition $\widehat{\mathbb{P}} = L_{K(h)} \bigoplus_{n\geq 0} \widehat{\mathbb{P}}_n$ with $\widehat{\mathbb{P}}_n M = L_{K(h)} M^{\otimes n}_{\mathrm{h}\Sigma_n}$. Write $\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ for the category of $K(h)$-local $\mathbb{E}_\infty$ algebras over E; we will abuse terminology and refer to these as *E-algebras*. We will write $\otimes$ for any of $\otimes_E$, $\otimes_{E_*}$, and $\otimes_{E_0}$, leaving which we mean to context.

Theorem 4.6 ([**Rez09**], [**Rez12**]). There is a monad $\mathbb{T}$ on the category of E_*-modules satisfying and determined by the following three items:

(1) The functor $\mathbb{T}$ preserves filtered colimits and reflexive coequalizers.

(2) There are natural maps $\mathbb{T}(M_*) \to \pi_*\widehat{\mathbb{P}}M$ for $M \in \mathcal{M}\mathrm{od}_E$ compatible with the monad structures on $\mathbb{T}$ and $\widehat{\mathbb{P}}$. In particular, the homotopy groups of any $A \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ naturally form a $\mathbb{T}$-algebra.

(3) There is a decomposition $\mathbb{T} \cong \bigoplus_{n\geq 0} \mathbb{T}_n$ compatible with the summands $\widehat{\mathbb{P}}_n \subset \widehat{\mathbb{P}}$, and if M is a finitely generated and free E-module then the map $\mathbb{T}_n(M_*) \to \pi_*\widehat{\mathbb{P}}_n M$ is an isomorphism.

In addition,

(4) $\mathbb{T}$ is an exponential monad, thus an E_*-plethory, with exponential structure inherited from the natural equivalences $\widehat{\mathbb{P}}_n(M \oplus N) \simeq \bigoplus_{i+j=n} \widehat{\mathbb{P}}_i M \otimes \widehat{\mathbb{P}}_j N$.

(5) $\mathbb{T}$ is an even-periodic plethory, with suspension maps inherited from the natural maps $\Sigma\widehat{\mathbb{P}}_n M \to \widehat{\mathbb{P}}_n \Sigma M$ defined for $n > 0$.

(6) $\mathbb{T}$ is smooth, in fact free, relative to alternating E_0-algebras (cf. Example 3.20).

Write $\Gamma = \Gamma(\mathbb{T})_{0,0} \subset \mathbb{T}(E_*)_0$ for the ordinary E_0-cobialgebroid underlying the even-periodic cobialgebroid $\Gamma(\mathbb{T})$.

(7) Let $\Gamma[n]$ denote the intersection of Γ with $\mathbb{T}(E_*)_{p^n}$. Then $\Gamma = \bigoplus_{n\geq 0} \Gamma[n]$ is a graded algebra. Moreover, this is a decomposition of coalgebras, and each $\Gamma[n]$ is finitely generated and free as a left E_0-module. In particular, Γ is a good E_0-cobialgebroid. Moreover, each $\Gamma[n]^\vee$ is a complete local ring with residue field κ.

(8) Let $\mathcal{X} = (\operatorname{Spec} E_0, \operatorname{Sp}^\vee \Gamma)$ be the formal category scheme associated to Γ, and let $\mathrm{Def} \subset \mathcal{X}$ be the full subcategory spanned by $\operatorname{Spf} E_0$. In other words, Def is the formal category scheme with objects $\operatorname{Spf} E_0$ and morphisms $\coprod_{n\geq 0} \operatorname{Spf} \Gamma[n]^\vee$, where $\Gamma[n]^\vee$ is given its adic topology. Consider Def as a presheaf of categories on the category of formal schemes Y such that $\mathcal{O}_Y$ is a complete local ring equipped with its adic topology. Then $\mathrm{Def}(Y)$ is the category with

   (a) Objects: Deformations of $\mathbb{G}_0$ to Y. These can be summarized as diagrams of Cartesian squares
$$\begin{tikzcd} \mathbb{G}_0 \arrow[d] & \mathbb{H}_0 \arrow[l, "\alpha"'] \arrow[r] \arrow[d] & \mathbb{H} \arrow[d] \\ X_0 & Y_0 \arrow[l, "j"'] \arrow[r] & Y \end{tikzcd},$$
   where $Y_0 = \operatorname{Spec}(\mathcal{O}_Y/\mathfrak{m}_Y) \subset Y$ is the special fiber of Y and α is a group homomorphism. We will write these as $\langle \mathbb{H}, j, \alpha\rangle$, or just $\mathbb{H}$ when the remaining structure is understood.

   (b) Morphisms: A morphism $f\colon \langle \mathbb{H}, j, \alpha\rangle \to \langle \mathbb{H}', j', \alpha'\rangle$ in $\mathrm{Def}(Y)$ classified by a map landing in the connected component $\operatorname{Spf} \Gamma[n]^\vee \subset \operatorname{Sp}^\vee \Gamma$ is a deformation of the n-fold Frobenius homomorphism of $\mathbb{G}_0$. These can be summarized as homomorphisms $f\colon \mathbb{H} \to \mathbb{H}'$ over Y such that the diagram

$$
\begin{tikzcd}
\mathbb{G}_0 \arrow[d] \arrow[dr, "F^n"] & & \mathbb{H}_0 \arrow[ll, "\alpha"'] \arrow[r] \arrow[d] \arrow[dr, "f_0"] & & \mathbb{H} \arrow[d] \arrow[dr, "f"] & \\
& \mathbb{G}_0 \arrow[d] & & \mathbb{H}'_0 \arrow[ll, "\alpha'"'] \arrow[rr] \arrow[d] & & \mathbb{H}' \arrow[d] \\
X_0 \arrow[dr, "\sigma^n"] & & Y_0 \arrow[ll, "j"'] \arrow[rr] \arrow[dr, "="] & & Y \arrow[dr, "="] & \\
& X_0 & & Y_0 \arrow[ll, "j'"'] \arrow[rr] & & Y
\end{tikzcd}
$$

commutes. Here, σ^n is the n-fold algebraic Frobenius and F^n is the n-fold absolute Frobenius homomorphism.

(9) Equivalently, $\operatorname{Spf}\Gamma[n]^\vee$ is the formal scheme $\mathrm{Sub}^n_{\mathbb{G}}$ classifying degree p^n subgroups of $\mathbb{G}$. The target map $t\colon \mathrm{Sub}^n_{\mathbb{G}} \to X$ sends a degree p^n subgroup $K \subset \mathbb{H}$, where $\mathbb{H}$ is a deformation of $\mathbb{G}_0$, to the quotient $\mathbb{H}/K$, considered as a deformation of $\mathbb{G}_0$ via the isomorphism $(\mathbb{H}/K)_0 \cong \mathbb{G}_0/(\mathbb{G}_0[F^n]) \cong (\sigma^n)^*\mathbb{G}_0$, where $\mathbb{G}_0[F^n]$ is the kernel of the n-fold Frobenius and the second equivalence is given by the n-fold relative Frobenius.

Now,

(10) Write $\omega = \pi_2 E$, so that $\omega = \omega_{\mathbb{G}}$ is the module of invariant differentials on $\mathbb{G}$. Then the Γ-module structure on ω, encoding what is necessary to pass from the ungraded cobialgebroid Γ to the even-periodic cobialgebroid $\Gamma(\mathbb{T})$, is given by the quasicoherent sheaf on Def sending a deformation $\mathbb{H}$ to the module of invariant differentials $\omega_{\mathbb{H}}$ on $\mathbb{H}$.

Slightly abusing notation, write $\mathrm{LMod}^{\heartsuit}_{\Gamma}$ for the category of graded modules over Γ, equivalent to the category of quasicoherent sheaves of graded modules on Def, and write $\mathcal{R}\mathrm{ing}^{\heartsuit}_{\Gamma}$ for the category of alternating Γ-rings, equivalent to the category of quasicoherent sheaves of alternating rings on Def.

(11) The restriction $\mathcal{R}\mathrm{ing}^{\heartsuit}_{\mathbb{T}} \to \mathcal{R}\mathrm{ing}^{\heartsuit}_{\Gamma}$ is fully faithful when restricted to the full subcategory of p-torsion free $\mathbb{T}$-rings. The essential image is spanned by those p-torsion free Γ-rings B whose underlying ungraded Γ-ring B_0 satisfies either of the following equivalent congruence criteria:

(a) Let $X_1 = \operatorname{Spf}(E_0/(p))$ and $\mathbb{G}_1 = X_1 \times_X \mathbb{G}$. Consider the map $X_1 \to \operatorname{Spf}\Gamma[1]^\vee \cong \mathrm{Sub}^1_{\mathbb{G}}$ classifying $\mathbb{G}_1[F]$, and choose a lift of this to an E_0-linear map $\Gamma[1]^\vee \to E_0$. Dualize this to obtain an element $Q \in \Gamma[1]$ which is well defined in $\Gamma[1]/(p)$. Then $Qx \equiv x^p \pmod p$ for all $x \in B_0$.

(b) Let $\mathcal{F}$ be the quasicoherent sheaf of rings associated to B_0. Then for every deformation $\mathbb{H}$ of $\mathbb{G}_0$ to Y with $p = 0$ in $\mathcal{O}_Y$, the diagram

$$\begin{tikzcd}
\mathcal{F}_Y(\sigma^*\mathbb{H}) \arrow[dd, "\simeq"] \arrow[dr, "\mathcal{F}_Y(F)"] & \\
& \mathcal{F}_Y(\mathbb{H}) \\
\sigma^*\mathcal{F}_Y(\mathbb{H}) \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

commutes. Here, $F\colon \mathbb{H} \to \sigma^*\mathbb{H}$ is the relative Frobenius on $\mathbb{H}$, the left vertical isomorphism arises from pseudonaturality of $\mathcal{F}$, and σ is the algebraic Frobenius on the $\mathcal{O}_Y$-ring $\mathcal{F}_Y(\mathbb{H})$.

Write $\Delta = \Delta(\mathbb{T})_{0,0}$ and, slightly abusing notation, write $\mathrm{L}\mathcal{M}\mathrm{od}_\Delta$ for the category of graded modules over Δ. So $\mathrm{L}\mathcal{M}\mathrm{od}_\Delta \simeq \mathrm{L}\mathcal{M}\mathrm{od}_\Gamma$ in the manner described in Proposition 4.5, and the choice of trivialization of $\omega_\mathbb{G}$ gives an isomorphism of algebras $\Delta \cong \Gamma$. Then

(12) The algebra Δ is graded compatibly with Γ, and both Γ and Δ are Koszul E_0-algebras. Moreover, $H^n(\Delta) = 0$ for $n > h$. In particular, every Δ-module which is projective over E_0 admits a length h projective Koszul resolution. □

Example 4.9. The fundamental example is given when $\kappa = \mathbb{F}_p$ and $\mathbb{G}_0$ is the formal multiplicative group. The associated Lubin-Tate spectrum $E = KU_p$ is the p-completion of complex K-theory. In this case the full subcategory of $\mathcal{R}\mathrm{ing}_\mathbb{T}^\heartsuit$ spanned by those objects concentrated in even degrees is equivalent to the category of θ-rings over $\mathbb{Z}_p$ (cf. Example 3.19).

The full category $\mathcal{R}\mathrm{ing}_\mathbb{T}^\heartsuit$ may be identified as follows. The $\Gamma = \mathbb{Z}_p[\psi]$-module $\omega = \pi_2 KU_p$ may be identified as $\mathbb{Z}_p\{\beta\}$ with action $\psi(\beta) = p\beta$. Following Remark 4.5, if $A \in \mathcal{R}\mathrm{ing}_\mathbb{T}^\heartsuit$ and $x, y \in A_{-1}$, then $\psi(xy) = p\psi(x)\psi(y) \in A_0$. In the generic case we may factor out this p, and in the end identify $\mathcal{R}\mathrm{ing}_\mathbb{T}^\heartsuit$ as the category of $\mathbb{Z}/(2)$-graded alternating rings A over $\mathbb{Z}_p$ equipped with a θ-ring structure on A_0 and an additive map $\psi\colon A_{-1} \to A_{-1}$, such that if $x \in A_0$ or $y \in A_0$, then $\psi(xy) = \psi(x)\psi(y)$, and if $x, y \in A_{-1}$, then $\theta(xy) = \psi(x)\psi(y)$.

See Remark 4.8 for a description of the general $K(1)$-local case. ◁

Example 4.10. Let C_0 be the elliptic curve over a perfect field κ of characteristic $p = 2$ with affine equation $v^2 + v = u^3$ and identity $(u, v) = (0, 0)$. This is a supersingular elliptic curve with formal group $\mathbb{G}_0$, whose universal deformation $\mathbb{G}$ can be identified as the formal group associated to the elliptic curve C over $R = W(\kappa)[[a]]$ with affine equation $v^2 + auv + v = u^3$ and identity $(u, v) = (0, 0)$; we choose u as our preferred coordinate for this formal group. The structure of power operations for the resulting Lubin-Tate spectrum have been calculated by Rezk [**Rez08**], and we have recalled the structure of the associated even-periodic cobialgebroid in Examples 3.14, 3.18, 4.6, and 4.7.

A congruence element of $\Gamma[1]$ allowing us to recover the full category of $\mathbb{T}$-rings is given by Q_0. Thus if A is a 2-torsion free Γ-ring, with Γ-ring structure on A_0 encoded by a map $P\colon A_0 \to A_0[d]/(d^3 = ad + 2)$, then A is the underlying Γ-ring of a $\mathbb{T}$-ring, necessarily uniquely, if and only if $P(x) \equiv x^2 \pmod{d}$ for all $x \in A_0$.

The operation Q_0 generically decomposes as $Q_0(x) = x^2 + 2\theta(x)$ for some $\theta \in \mathbb{T}(E_*)_0$, and $\mathbb{T}(E_*)_0$ is a polynomial ring on certain iterates of θ, Q_1, and Q_2. The algebra $\Delta = Q(\mathbb{T}(E_*)_0)$ is generated by θ, Q_1, and Q_2, subject those relations seen in Γ among Q_1 and Q_2, as well as

$$\begin{aligned}\theta a &= a^2\theta - aQ_1 + 3Q_2\\ Q_1\theta &= Q_2Q_1 - 2\theta Q_2,\\ Q_2\theta &= \theta Q_1 + a\theta Q_2 - Q_1Q_2.\end{aligned}$$

The composite $\Gamma \to \mathbb{T}(E_*)_0 \to \Delta$ is

$$Q_0 \mapsto 2\theta, \qquad Q_1 \mapsto Q_1, \qquad Q_2 \mapsto Q_2.$$

The suspension isomorphism $\Delta \to \Gamma$ is

$$\theta \mapsto -Q_2, \qquad Q_1 \mapsto -Q_0 - aQ_2, \qquad Q_2 \mapsto -Q_1.$$

If M is a Γ-module encoded by a coaction $P\colon M \to \Gamma[1]^\vee{}_s\!\otimes M$, then $\omega \otimes M = M$ as R-modules, with Γ-module structure encoded by $-dP\colon M \to \Gamma[1]^\vee{}_s\!\otimes M$. If A is an augmented $\mathbb{T}$-ring, then $Q(A)$ inherits the structure of a Γ-module, and the Frobenius congruence implies that the image of $P\colon Q(A) \to \Gamma[1]^\vee{}_s\!\otimes Q(A)$ is divisible by d. If $Q(A)$ is torsion-free, then $\frac{-1}{d}P\colon Q(A) \to \Gamma[1]^\vee{}_s\!\otimes Q(A)$ defines a Γ-module, written $\omega^{-1/2} \otimes M$. To be precise, the underlying graded R-module of $\omega^{-1/2} \otimes M$ differs from M by a shift in degrees. With these definitions, $\omega^{-1/2} \otimes M$ is a model for the image of the Δ-module $Q(A)$ under the Morita equivalence $\mathrm{LMod}_\Delta \simeq \mathrm{LMod}_\Gamma$ of Proposition 4.5. Similar remarks are available for arbitrary Lubin-Tate spectra. $\triangleleft$

REMARK 4.7. Suppose that $\mathbb{G}_0$ is a formal group of height 2. Then the following description of $H^*(\Gamma)$ is given in [**Rez13**]. First, there is a commutative diagram

$$\begin{tikzcd} E_0 \arrow[r, "t"] \arrow[d, "t"] \arrow[dd, bend right, "\Psi"'] & \Gamma[1]^\vee \arrow[d, "\Gamma[1]^\vee\otimes t"] \\ \Gamma[2]^\vee \arrow[r, "c"] \arrow[d, "q"] & \Gamma[1]^\vee{}_s\!\otimes_t \Gamma[1]^\vee \arrow[d, "f"] \\ E_0 \arrow[r, "s"] & \Gamma[1]^\vee \end{tikzcd},$$

the bottom square of which is Cartesian. Here, s, t, and c are part of the structure of the cobialgebroid Γ. As $\Gamma[1]^\vee$ classifies rank p subgroups H, the tensor product $\Gamma[1]^\vee{}_s\!\otimes_t \Gamma[1]^\vee$

classifies chains $H_0 \subset H_1$ where both H_0 and H_1/H_0 are of rank p. It is the remaining maps which are special to height 2; the map q classifies the p-torsion subgroup $\mathbb{G}[p]$ and the map f classifies the chain $H \subset \mathbb{G}[p]$, where H is the universal rank p subgroup defined over $\Gamma[1]^\vee$.

In particular, Ψ is the automorphism of E_0 classifying the deformation $\mathbb{G}/\mathbb{G}[p]$. For the Lubin-Tate spectrum of Example 4.10, this square is described explicitly in Example 4.7; in this example Ψ is the identity provided $\kappa \subset \mathbb{F}_4$.

Quadraticity of Γ implies that $H^0(\Gamma) = E_0$ and $H^1(\Gamma) = \Gamma[1]^\vee$. As the bottom square of the above diagram is Cartesian, we may moreover identify

$$H^2(\Gamma) = \mathrm{Coker}(c\colon \Gamma[2]^\vee \to \Gamma[1]^\vee {}_s\!\otimes_t \Gamma[1]^\vee) \cong \mathrm{Coker}(s\colon E_0 \to \Gamma[1]^\vee).$$

All higher cohomology groups vanish. Multiplication $H^1(\Gamma) \otimes H^1(\Gamma) \to H^2(\Gamma)$ is the composite

$$\Gamma[1]^\vee {}_s\!\otimes_t \Gamma[1]^\vee \to \Gamma[1]^\vee \to \Gamma[1]^\vee / s(E_0) \cong H^2(\Gamma),$$

where the first map is f. The right E_0-module structure on $H^2(\Gamma)$ is through s, and the left E_0-module structure twists this by Ψ, i.e. $a \cdot x = x \cdot s(\Psi(a))$ for $a \in E_0$ and $x \in H^2(\Gamma)$.

Koszul complexes computing Ext_Γ are readily obtained from this; see Example 4.15 for an explicit example. $\triangleleft$

We would like to apply our understanding of algebraic structures such as $\mathbb{T}$ to obstruction-theoretic machinery for computing with E-algebras. Here, one runs into the subtlety that $\mathbb{T}$ does not perfectly encode the structure of all operations that act on the homotopy groups of E-algebras: the map $\mathbb{T}(\pi_* F) \to \pi_* \widehat{\mathbb{P}} F$ is not an isomorphism for $F \in \mathcal{M}\mathrm{od}_E^{\mathrm{free}}$. The missing piece is that the $K(h)$-local condition on our E-algebras enforces a certain completeness condition on their homotopy groups, and this is not seen by $\mathbb{T}$.

Write $\mathcal{A}_\mathfrak{m}$ for the 0'th left-derived functor of $\mathfrak{m}$-adic completion on $\mathcal{M}\mathrm{od}_{E_*}^\heartsuit$. This is a localization, and we will call the $\mathcal{A}_\mathfrak{m}$-local objects $\mathfrak{m}$-*complete*, and denote the category of $\mathcal{A}_\mathfrak{m}$-local objects by $\mathcal{M}\mathrm{od}_{E_*}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$. Although this is distinct from the classic notion of $\mathfrak{m}$-adic completeness, we will not use the classic notion, and so minimal confusion should arise. The functor $\mathcal{A}_\mathfrak{m}$ has been studied in [**HS99**, Appendix A] under the name of L-completion, in [**Rez18**] under the name of analytic completion, and in other places by other names; when $\mathfrak{m} = (p)$, this is Ext-p-completion in the sense of [**BK72a**, Section VI.2.1], as we have encountered in Example 3.9. These concepts are a truncation of those reviewed in Section 2.5, and we will recall what we need in Subsection 4.2.4 below.

The main theorem of [**BF15**] gives $\mathcal{A}_\mathfrak{m}\mathbb{T}$ the structure of a monad; we will not use this theorem, but instead show how it follows easily from the use of algebraic theories and the general philosophy that constructing the category of algebras for a monad can be easier than constructing the monad itself. Write $\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}$ for the category of ($K(h)$-local $\mathbb{E}_\infty$)

E-algebras which are free on a free E-module. Then $\mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}$ is a discrete theory whose category of discrete models is monadic over $\mathcal{M}\mathrm{od}_{E_*}^{\heartsuit}$; write the associated monad as $\widehat{\mathbb{T}}$. The general properties of this construction are as described in Proposition 2.8.

Proposition 4.8. The forgetful functor $\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}} \to \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\heartsuit}$ is fully faithful, with essential image spanned by those $\mathbb{T}$-rings whose underlying E_*-module is $\mathfrak{m}$-complete. In particular, $\widehat{\mathbb{T}}$ is a plethory for the theory of $\mathfrak{m}$-complete E_*-modules.

Proof. There is by construction a map $\mathbb{T} \to \widehat{\mathbb{T}}$ of monads on $\mathcal{M}\mathrm{od}_{E_*}^{\heartsuit}$. As $\widehat{\mathbb{T}}$ takes values in $\mathfrak{m}$-complete modules, as a map of functors this factors as $\mathbb{T} \to \mathcal{A}_{\mathfrak{m}}\mathbb{T} \to \widehat{\mathbb{T}}$. By Lemma 2.14 and Lemma 2.15, it is sufficient to verify that $\mathcal{A}_{\mathfrak{m}}\mathbb{T} \to \widehat{\mathbb{T}}$ is an isomorphism of functors. As both source and target preserve geometric realizations, it is sufficient to verify that $\mathcal{A}_{\mathfrak{m}}\mathbb{T}(F_*) \to \widehat{\mathbb{T}}(F_*)$ is an isomorphism when F_* is a free E_*-module. Fix such F_*, and write $F_* = \pi_* F$ for a free E-module F. By the construction of $\mathbb{T}$ and $\widehat{\mathbb{T}}$, there is a commutative diagram

$$\begin{tikzcd}
\mathbb{T}(F_*) \arrow[r] \arrow[d] & \mathcal{A}_{\mathfrak{m}}\mathbb{T}(F_*) \arrow[r] \arrow[d, "\cong"] & \widehat{\mathbb{T}}(F_*) \arrow[d, "\cong"] \\
\pi_*\widehat{\mathbb{P}}F \arrow[r, "\cong"] & \pi_*\widehat{\mathbb{P}}F \arrow[r, "\cong"] & \pi_*\widehat{\mathbb{P}}F
\end{tikzcd}.$$

Here, the right vertical map is an isomorphism by construction, and the middle vertical map is an isomorphism as $\mathbb{T}(F_*)$ is free [**Rez09**, Proposition 4.17]. Thus the top right horizontal map is an isomorphism, and this proves the proposition. □

Remark 4.8. The abstract construction of $\widehat{\mathbb{T}}$ certainly does not rely on E being a Lubin-Tate spectrum, and with some work the algebraic constructions of Theorem 4.6 can also be extended to more general $K(h)$-local even-periodic $\mathbb{E}_\infty$ ring spectra. To set this up correctly would take us too far afield, so we will not do so here. However, let us note how the algebraic story plays out at height $h = 1$.

As discussed in [**Hop14**], the transfer yields an equivalence $L_{K(1)}\Sigma^\infty B\Sigma_p \simeq \mathbb{S}_{K(1)}$, using this one can define an operation $\theta \in \pi_0 L_{K(1)}\Sigma^\infty_+ B\Sigma_p$ making π_0 of an arbitrary $K(1)$-local $\mathbb{E}_\infty$ ring spectrum into a θ-ring, and in fact if R is a $K(1)$-local $\mathbb{E}_\infty$ ring, then $\pi_0 L_{K(1)}\mathbb{P}_R R$ is the free Ext-p-complete θ-ring on $\pi_0 R$. If R is even-periodic, then this splitting and identification extends to nonzero degrees. It follows that $\mathrm{h}\mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,free}}$ is a theory of Ext-p-complete $\mathbb{Z}/(2)$-graded θ-rings equipped with a map from $R_\star$.

To be precise, the correct notion of a "$\mathbb{Z}/(2)$-graded θ-ring" must incorporate the $\mathbb{Z}_p[\psi]$-module structure on $\omega = \pi_2 R$, in the same manner as it was incorporated in Example 4.9. This plays out as follows. Under the suspension map $R_0^\wedge \mathbb{P}_p\mathbb{S} \to R_2^\wedge \mathbb{P}_p\Sigma^2\mathbb{S}$, the operation θ is sent to some additive operation $\frac{1}{p}\psi\colon \pi_2 \to \pi_2$. Now the category of models of $\mathrm{h}\mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,free}}$

is equivalent to the category of pairs (A_0, A_{-1}) where: first, A_0 is an Ext-p-complete θ-ring under R_0; second, A_{-1} is an Ext-p-complete A_0-module and $\mathbb{Z}[\psi]$-module satisfying $\psi(a_0 \cdot a_{-1}) = \psi(a_0) \cdot \psi(a_{-1})$ for $a_0 \in A_0$ and $a_{-1} \in A_{-1}$, where ψ is the operation on A_0 defined by $\psi(a_0) = a_0^p + p\theta(a_0)$; and third, there is a suitably alternating and associative multiplication $m\colon \omega \otimes A_{-1} \otimes A_{-1} \to A_0$ satisfying $\theta(m(u \otimes a \otimes a')) = m(\frac{1}{p}\psi(u) \otimes \psi(a) \otimes \psi(a'))$. $\triangleleft$

We end this subsection by pointing to where one can find some computations of the structure of E-power operations. The height $h = 1$ case is as covered in Remark 4.8, and explicit computations at heights $h \geq 3$ are not currently feasible, so we are left with height $h = 2$, where computations are made possible by the theory of elliptic curves.

The first full explicit computation in this setting is the computation at $p = 2$ of Rezk [**Rez08**] recalled in Example 4.10. Further computations at $p = 2$ have been carried out by Schumann [**Sch14**], allowing for elliptic curves with any Weierstraß equation of the form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x$, i.e. with $a_6 = 0$. Notably, this work gives a closed-form description of the total power operation on $E^0BU(1) \cong E_0[[u]]$; for the Lubin-Tate spectrum of Example 4.10, this is the ring map

$$P\colon E_0[[u]] \to E_0[[u]][d]/(d^3 - ad - 2), \qquad P(u) = \frac{u^2 - du}{1 + d^2u}.$$

At $p = 3$, for Lubin-Tate spectra E associated to certain elliptic curves, E-power operations have been computed by Nendorf [**Nen12**] and by Zhu [**Zhu14**]. The latter also discusses the power operation structure on $L_{K(1)}E$ for the height $h = 2$ Lubin-Tate spectrum E in question. Further work of Zhu in [**Zhu19**] gives a recipe that works for arbitrary primes.

We point also to [**Rez13**], which contains a wealth of information at heights $h \leq 2$, and in particular a number of computations in the cohomology of $\mathbb{T}$-rings at heights $h \leq 2$.

4.2.4. Completions. Fix notation as in the preceding section. Following Proposition 4.8, we are interested in the homotopy theory of certain completed contexts. We studied some of the general interaction between theories and completions in Section 2.5; here we explain how things fit together in the context of $\mathbb{T}$-rings.

Let us first recall the definitions in our particular context. Given $M \in \mathcal{M}\mathrm{od}_{E_*}$, say that M is $\mathfrak{m}$-*nilpotent* if $M[x^{-1}] = 0$ for all $x \in \mathfrak{m}$, is $\mathfrak{m}$-*local* if $\mathrm{Map}(N, M) \simeq *$ for all $\mathfrak{m}$-nilpotent N, and is $\mathfrak{m}$-*complete* if $\mathrm{Map}(N, M) \simeq *$ for all $\mathfrak{m}$-local N. The full subcategory $\mathcal{M}\mathrm{od}_{E_*}^{\mathrm{Cpl}(\mathfrak{m})} \subset \mathcal{M}\mathrm{od}_{E_*}$ of $\mathfrak{m}$-complete modules is a reflective subcategory, and an explicit formula for the reflection $(-)^\wedge_\mathfrak{m}$ is given as follows. Choose generators $u_0, \ldots, u_{h-1} \in \mathfrak{m}$ and fix $M \in \mathcal{M}\mathrm{od}_{E_*}$. Then $M^\wedge_\mathfrak{m}$ is the total cofiber of the h-cube obtained as the external product of the 1-cubes $T_i - u_i\colon M[[T_i]] \to M[[T_i]]$. If $M \in \mathcal{M}\mathrm{od}_{E_*}^\heartsuit$, then $\mathcal{A}_\mathfrak{m}M = \pi_0(M^\wedge_\mathfrak{m})$.

Observe that the preceding definitions can be applied equally well in any linear setting in which there is a natural action by the elements of $\mathfrak{m}$. In particular, they apply to $\mathcal{M}\mathrm{od}_E$, where $\mathfrak{m}$-completion coincides with $K(h)$-localization. In general, if $\mathcal{M}$ is some category over $\mathcal{M}\mathrm{od}_E$ or $\mathcal{M}\mathrm{od}_{E_*}$, we will write $\mathcal{M}^{\mathrm{Cpl}(\mathfrak{m})} \subset \mathcal{M}$ for the full subcategory spanned by those objects whose underlying module is $\mathfrak{m}$-complete. Thus for instance Proposition 4.8 tells us that

$$\mathcal{M}\mathrm{odel}^{\heartsuit}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}} \simeq \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\heartsuit,\mathrm{Cpl}(\mathfrak{m})}.$$

The key fact that allows us to handle completions is that this continues to hold at the level of simplicial rings.

Theorem 4.7. There is an equivalence $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_E^{\mathrm{loc,free}}} \simeq \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m})}$.

Proof. This follows from Proposition 2.17 and Proposition 4.8, as $\mathcal{A}_{\mathfrak{m}}\mathbb{T}(F_*) \simeq \mathbb{T}(F_*)^\wedge_{\mathfrak{m}}$ for $F_* \in \mathcal{M}\mathrm{od}_{E_*}^{\mathrm{free}}$ by tameness of $\mathbb{T}(F_*)$. $\square$

In particular, given $R \in \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, $S \in \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, $M \in \mathcal{A}\mathrm{b}(\mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}/S}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}) \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, and $A \in \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}/S}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, all the following spaces are equivalent:

$$\begin{aligned}\mathrm{Map}_{R/\mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m})}/S}(A, S \ltimes B^n M) &\simeq \mathrm{Map}_{R\otimes\mathbb{T}/S}(A, S \ltimes B^n M) \simeq \mathcal{H}^n_{R\otimes\mathbb{T}/S}(A; M) \\ &\simeq \mathcal{E}\mathrm{xt}^n_{S\otimes\Delta}(S \otimes^{\mathbb{L}}_A \Omega_{A|R}, M) \simeq \mathcal{E}\mathrm{xt}^n_{S\otimes\Delta}(S \mathbin{\widehat{\otimes}}^{\mathbb{L}}_A \mathbb{L}\widehat{\Omega}_{A|R}, M).\end{aligned}$$

Because of this, we will generally write things in terms of $\mathbb{T}$, although there is a sense in which $\widehat{\mathbb{T}}$ is more fundamental in our setting.

The primary subtlety of completions relevant to us is that Theorem 4.7 does not extend to all settings. For example, if R is an E_*-ring, then there is a category $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m})}$ of $\mathfrak{m}$-complete R-modules, and $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m})} \simeq \mathrm{L}\mathcal{M}\mathrm{od}_{\mathcal{P}}$ where $\mathcal{P} \subset \mathcal{M}\mathrm{od}_R$ is the full subcategory spanned by the $\mathfrak{m}$-completions of free R-modules. But the failure of coproducts to be exact in general [**Bak09**, Appendix B] can force this theory $\mathcal{P}$ to be non-discrete, and in particular $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m})}$ need not be the derived category of $\mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$.

Recall that M is *tame* if $(M^{\oplus I})^\wedge_{\mathfrak{m}}$ is discrete for any set I; it is sufficient to consider the case $I = \omega$. Then most issues with completions vanish so long as we build on tame objects. For example, if $R \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ with R_* tame, then $\pi_*\widehat{\mathbb{P}}_R(R \mathbin{\widehat{\otimes}} F) \simeq \mathcal{A}_{\mathfrak{m}}(R_* \otimes \mathbb{T}(F_*))$ for $F \in \mathcal{M}\mathrm{od}_E^{\mathrm{free}}$, and there is an equivalence $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,free}}} \simeq \mathcal{R}\mathrm{ing}_{R_*\otimes\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m})}$.

We end by noting the following.

Lemma 4.6. Fix $R \in \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, $S \in \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}}^{\heartsuit}$, $M \in \mathrm{L}\mathcal{M}\mathrm{od}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, and $A \in \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}/S}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$. If A is smooth as an $\mathfrak{m}$-complete alternating R-ring in the sense that $\mathbb{L}\widehat{\Omega}_{A_*|R_*}$ is the completion of a projective A_*-module, then $H^n_{R\otimes\mathbb{T}/S}(A; M) = 0$ for $n > h$.

Proof. More generally, choose $N \in \mathrm{LMod}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m})}$ whose underlying S-module is the completion of a projective S-module; we claim that $\mathrm{Ext}^n_{S\otimes\Delta}(N,M)=0$ for $n>h$. In the case where $S=E_*$ and N is the completion of a Δ-module whose underlying E_*-module is projective, this is a consequence of part (12) of Theorem 4.6. In general, consider the diagram

$$\begin{tikzcd}
\mathrm{LMod}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit} \arrow[r] \arrow[d] & \mathcal{M}\mathrm{od}_S^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit} \arrow[d] \\
\mathrm{LMod}_{S\otimes\Delta}^{\heartsuit} \arrow[r] \arrow[d] & \mathcal{M}\mathrm{od}_S^{\heartsuit} \arrow[d] \\
\mathrm{LMod}_{\Delta}^{\heartsuit} \arrow[r] & \mathcal{M}\mathrm{od}_{E_*}^{\heartsuit}
\end{tikzcd}.$$

Write $\mathrm{LMod}_{S\otimes\Delta}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit} \simeq \mathrm{LMod}_F^{\heartsuit}$, where F is an algebra over $\mathcal{M}\mathrm{od}_S^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$. Each square in the above is distributive, so by Lemma 3.12 the algebra F is Koszul, with length h Koszul resolutions. Though N may not be discrete if S is not tame, we may nonetheless apply Lemma 3.7 to identify $\mathcal{E}\mathrm{xt}_{S\otimes\Delta}(N,M)$ as the totalization of $\mathcal{E}\mathrm{xt}_{S\otimes\Delta}(B(S\otimes\Delta,S\otimes\Delta,N),M)$. As M is $\mathfrak{m}$-complete and discrete, there is an equivalence $B_{S\otimes\Delta}(N,M)\simeq B_F(\pi_0 N,M)$. As $\pi_0 N$ is a projective object of $\mathcal{M}\mathrm{od}_S^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, there is a quasiisomorphism $B_F(\pi_0N,M)\simeq K_F(\pi_0 N,M)$ by Koszulity. As $K_F(\pi_0N,M)$ is a length h complex, this proves the lemma. $\square$

4.2.5. Mapping spaces and highly structured orientations. We can now describe some applications of the preceding theory. Fix notation as in the preceding subsections; in particular E is a Lubin-Tate spectrum of height h.

Theorem 4.8. Fix $R \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$, and choose $S\in\mathcal{C}\mathrm{Alg}_R^{\mathrm{loc}}$ such that $R_*\to S_*$ is surjective (such as $S=0$ or $S=R$). Fix $A,B\in\mathcal{C}\mathrm{Alg}_{R/S}^{\mathrm{loc}}$, and choose a map $\phi\colon A_*\to B_*$ in $\mathcal{R}\mathrm{ing}_{R_*\otimes\mathbb{T}/S_*}$. Let $\mathcal{C}\mathrm{Alg}_{R/S}^{\phi}(A,B)$ be the space of lifts of ϕ to a map in $\mathcal{C}\mathrm{Alg}_{R/S}$. Then there is a decomposition

$$\mathcal{C}\mathrm{Alg}_{R/S}^{\phi}(A,B)\simeq\lim_{n\to\infty}\mathcal{C}\mathrm{Alg}_{R/S}^{\phi,\leq n}(A,B),$$

with layers fitting into fiber sequences

$$\mathcal{C}\mathrm{Alg}_{R/S}^{\phi,\leq n}(A,B)\to\mathcal{C}\mathrm{Alg}_{R/S}^{\phi,\leq n-1}(A,B)\to\mathcal{H}^{n+1}_{R_*\otimes\mathbb{T}/B_*}(A_*;\pi_*\Omega^n F),$$

where $F=\mathrm{Fib}(B\to S)$. In particular,

(1) There are successively defined obstructions in $H^{n+1}_{R_*\otimes\mathbb{T}/B_*}(A_*;\pi_*\Omega^nF)$ for $n\geq 1$ to exhibiting a point of $\mathcal{C}\mathrm{Alg}_{R/S}^{\phi}(A,B)$;

(2) Once a point of $\mathcal{C}\mathrm{Alg}_{R/S}^{\phi}(A,B)$ is chosen, there is a fringed spectral sequence of signature

$$E_1^{p,q}=H^{p-q}_{R_*\otimes\mathbb{T}/B_*}(A_*;\pi_*\Omega^pF)\Rightarrow\pi_q(\mathcal{C}\mathrm{Alg}_{R/S}(A,B),f),\quad d_r^{p,q}\colon E_r^{p,q}\to E_r^{p+r,q-1}.$$

Specializing further, if A_* is smooth as an $\mathfrak{m}$-complete alternating R_*-ring, then

(3) If $h = 1$, then $\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$ is nonempty. Moreover,

$$\pi_0 \mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B) \cong H^1_{R_* \otimes \mathbb{T}/B_*}(A_*; \pi_* \Omega F),$$

and if we choose $f \in \mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$, then there are short exact sequences

$$\begin{aligned} 0 \to H^1_{R_* \otimes \mathbb{T}/B_*}(A_*; \pi_* \Omega^{n+1} F) &\to \pi_n(\mathcal{C}\mathrm{Alg}_{R/S}(A, B), f) \\ &\to H^0_{R_* \otimes \mathbb{T}/B_*}(A_*; \pi_* \Omega^n F) \to 0 \end{aligned}$$

for $n \geq 1$.

(4) If $h = 1$ and each of the rings in question is concentrated in even degrees, then $\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$ is connected. Moreover,

$$\pi_{2n-\epsilon} \mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B) \cong H^{\epsilon}_{R_* \otimes \mathbb{T}/B_*}(A_*; \pi_* \Omega^{2n} F)$$

for $n \geq 1$ and $\epsilon \in \{0, 1\}$.

(5) If $h = 2$ and each of the rings in question is concentrated in even degrees, then $\mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$ is nonempty. Moreover, $\pi_0 \mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B) \cong H^2_{R_* \otimes \mathbb{T}/B_*}(A_*; \pi_* \Omega^2 F)$, and if we choose $f \in \mathcal{C}\mathrm{Alg}^{\phi}_{R/S}(A, B)$, then there are short exact sequences

$$\begin{aligned} 0 \to H^2_{R_* \otimes \mathbb{T}/B_*}(A_*; \pi_* \Omega^{2(n+1)} F) &\to \pi_{2n}(\mathcal{C}\mathrm{Alg}_{R/S}(A, B), f) \\ &\to H^0_{R * \otimes \mathbb{T}/B_*}(A_*; \pi_* \Omega^{2n} F) \to 0 \end{aligned}$$

and isomorphisms

$$\pi_{2n-1}(\mathcal{C}\mathrm{Alg}_{R/S}(A, B), f) \cong H^1_{R_* \otimes \mathbb{T}/B_*}(A_*; \pi_* \Omega^{2n} F)$$

for $n \geq 1$.

Proof. The obstruction theory is an application of Theorem 2.11. The final statements then follow using Lemma 4.6. ◻

Remark 4.9. Following Remark 4.8, the preceding theorem applies when E is instead taken to be an arbitrary $K(1)$-local even-periodic $\mathbb{E}_\infty$ ring spectrum. ◁

Our main application of Theorem 4.8 is to the theory of $\mathbb{E}_\infty$ orientations. We first recall some history. Power operations for Lubin-Tate spectra were first studied by Ando [**And95**] precisely in the context of producing highly structured complex orientations. In particular, there it is shown that the Honda formal group law refines to a unique $\mathbb{H}_\infty$ orientation of its associated Lubin-Tate spectrum. The characterization of $\mathbb{H}_\infty$ orientations is described in a more general setting in Ando-Hopkins-Strickland [**AHS04**], which in addition transitions to explicitly considering MUP orientations, where MUP is the Thom spectrum of the tautological bundle over $\mathbb{Z} \times BU$. In brief, homotopy ring maps $MUP \to E$ correspond to coordinates on $\mathbb{G}$, and the conditions necessary for this coordinate to correspond to an $\mathbb{H}_\infty$-ring map can be described algebraically; we will call these coordinates *norm-coherent*, and will very briefly review the characterization in the proof of Theorem 4.9. Work of Zhu [**Zhu20**] extends the existence and uniqueness of norm-coherent coordinates to an arbitrary Lubin-Tate spectrum so long as $\mathbb{F}_p \subset \kappa$ is algebraic; in our language, this work says that the first map in

$$\mathcal{R}\mathrm{ing}_{\mathbb{T}}(E_0^\wedge MUP, E_0) \to \mathcal{R}\mathrm{ing}_{E_0}(E_0^\wedge MUP, \kappa) \cong \mathrm{Coord}(\mathbb{G}_0)$$

is an isomorphism. Recall that at height $h = 1$, the category of even $\mathbb{T}$-rings is exactly the category of θ-rings sliced under E_0. Here it is classical that $E_0 = W(\kappa)$ is in fact the cofree θ-ring on the E_0-ring κ, and so the above isomorphism is immediate, and $\mathbb{F}_p \subset \kappa$ need not be algebraic. An unpublished theorem of Rezk extends this to arbitrary heights, showing that E_0 is always the cofree $\mathbb{T}$-ring on the E_0-ring κ.

Given the preceding, we can safely say that $\mathbb{H}_\infty$ orientations are well-understood. By contrast, significantly less is known about $\mathbb{E}_\infty$ orientations. The exception to this is $\mathbb{E}_\infty$ orientations by MU at height $h = 1$; the case of p-adic K-theory has been studied by Walker [**Wal09**], and the more general $K(1)$-local case by Möllers [**Möl11**], using methods similar to those employed in [**AHR10**]; in short, in the $K(1)$-local context, every $\mathbb{H}_\infty$ orientation refines uniquely to an $\mathbb{E}_\infty$ orientation. In Hopkins-Lawson [**HL18**], a general obstruction theory for $\mathbb{E}_\infty$ orientations by MU is constructed that recovers the known $h = 1$ story. Even less is known about $\mathbb{E}_\infty$ orientations by MUP. The only work in this direction we are aware of is [**HY20**], which demonstrates their existence when $h = 1$ and $\kappa = \mathbb{F}_2$. Our contribution to this story is the following.

Theorem 4.9.

(1) Let R be a $K(1)$-local even-periodic $\mathbb{E}_\infty$ ring spectrum. Then every norm-coherent coordinate on the formal group associated to R refines uniquely to an $\mathbb{E}_\infty$ orientation $MUP \to R$.
(2) The multiplicative formal group law $x + y - xy$ refines uniquely to an $\mathbb{E}_\infty$ orientation $MUP \to KU$.
(3) Let E be a Lubin-Tate spectrum at height $h = 2$. Then every norm-coherent coordinate on $\mathbb{G}$ refines to an $\mathbb{E}_\infty$ orientation $MUP \to E$.

Proof. Claims (1) and (3) are immediate consequences of Theorem 4.8, as E_0MUP is smooth. Claim (2) follows directly from (1), the arithmetic fracture square, and the fact that $x+y-xy$ is a norm-coherent coordinate of the multiplicative formal group at all primes;

for completeness we give the details. First, arithmetic fracture gives a Cartesian square of the form

$$\begin{array}{ccc} \mathcal{C}\mathrm{Alg}(MUP, KU) & \longrightarrow & \prod_p \mathcal{C}\mathrm{Alg}_{KU_p}(KU_p \mathbin{\widehat{\otimes}} MUP, KU_p) \\ \downarrow & & \downarrow \\ \mathcal{C}\mathrm{Alg}(MUP_{\mathbb{Q}}, KU_{\mathbb{Q}}) & \longrightarrow & \mathcal{C}\mathrm{Alg}(MUP_{\mathbb{Q}}, (\prod_p KU_p)_{\mathbb{Q}}) \end{array}.$$

As $MUP_{\mathbb{Q}}$ is free as a rational $\mathbb{E}_\infty$ ring, the coordinate $x+y-xy$ gives points of the bottom two spaces, and $\pi_1 \mathcal{C}\mathrm{Alg}(MUP_{\mathbb{Q}}, (\prod_p KU_p)_{\mathbb{Q}}) = 0$. So it is sufficient to verify that for each prime p, the homotopy orientation associated to the formal group law $x+y-xy$ refines to a map $KU_p \mathbin{\widehat{\otimes}} MUP \to KU_p$ of KU_p-algebras. By Theorem 4.8, it is sufficient to verify that the coordinate associated to the formal group law $x+y-xy$ is norm-coherent at all primes.

The description of norm-coherent coordinates given in [**AHS04**, Section 4], in the case of a Lubin-Tate spectrum E, can be summarized as follows. A norm-coherent coordinate x on $\mathbb{G}$ is a coordinate such that for every formal scheme Y, map $f\colon Y \to X$, and finite subgroup $K \subset f^*\mathbb{G}$, we have $N_\pi \mu^* f^*(x) = q^* \alpha^*(x)$, where $\pi\colon K \times_Y f^*\mathbb{G} \to f^*\mathbb{G}$ is the projection, N_π is the associated norm map, $\mu\colon K \times_Y f^*\mathbb{G} \to f^*\mathbb{G}$ is the multiplication, and $\alpha\colon (f^*\mathbb{G})/K \to \mathbb{G}$ identifies $(f^*\mathbb{G})/K$ as a deformation of $\mathbb{G}_0$. To be precise, [**AHS04**] works in the context of level structures rather than finite subgroups, but the translation follows from the fact that $\coprod_{|A|=p^n} \mathrm{Level}(A, \mathbb{G}) \to \mathrm{Sub}^n_{\mathbb{G}}$ is surjective [**Str97**, Theorem 12.4]. In addition, it is sufficient to restrict to the case where K is a subgroup of rank p.

When $h = 1$, the kernel of formal multiplication by p is the unique subgroup of rank p. The p-series associated to the formal group law $x+y-xy$ is given by $[p](x) = 1-(1-x)^p$, so the above condition translates to asking that multiplication by $x+y-xy$ on the free $\mathbb{Z}_p[[x]]$-module $\mathbb{Z}_p[[x,y]]/(1-(1-y)^p)$ has determinant $1-(1-x)^p$. This itself can be checked by direct calculation, proving the theorem. $\square$

Remark 4.10. We do not know whether the uniqueness statement of Theorem 4.9 can be extended to height $h = 2$. This is equivalent to whether $\mathrm{Ext}^2_\Delta(\widehat{Q}(E_0^\wedge MUP), \omega) = 0$.

Remark 4.11. At height $h = 1$, in [**Möl11**, Corollary 3.13] it is shown that the choice of an orientation $MU \to E$ gives a weak equivalence $\mathcal{C}\mathrm{Alg}(MU, E) \simeq \mathrm{Map}(KU_p, E)$. This is reflected in the algebra of power operations by the following: there is an isomorphism

$$\widehat{Q}(E_0^\wedge BU) \cong \Delta \mathbin{\widehat{\otimes}} E_0^\wedge KU_p.$$

The inclusion $E_0^\wedge KU_p \to \widehat{Q}(E_0^\wedge BU)$ which extends by Δ-linearity to this isomorphism may be obtained from the map $KU_p \to L_{K(1)}\Sigma^\infty_+ BU$, itself obtained by applying the Bousfield-Kuhn functor [**Kuh08**] to the unit $BU \to \Omega^\infty\Sigma^\infty_+\Omega^\infty BU$. This map is special to height 1, and indeed $\widehat{Q}(E_0^\wedge BU)$ is no longer projective over Δ at higher heights (Example 4.14). $\triangleleft$

We now give a few more examples illustrating Theorem 4.8.

Example 4.11. In Chatham-Hahn-Yuan [**CHY19**], an interesting family of $\mathbb{E}_\infty$ ring spectra R_{h-1} at chromatic height h are constructed, and left open is the question of whether there exists an $\mathbb{E}_\infty$ map $R_{h-1} \to E$, where E is a Lubin-Tate spectrum of height h. Combining [**CHY19**, Theorem 7.6] with the preceding, we learn that there are $\mathbb{E}_\infty$ maps $R_1 \to E$ whenever E is a Lubin-Tate spectrum of height 2 associated to a supersingular elliptic curve. $\triangleleft$

Example 4.12. Given an arbitrary $\mathbb{E}_\infty$ ring spectrum R, we may define

$$\mathbb{A}^1(R) = \mathcal{C}\mathrm{Alg}(\Sigma^\infty_+\mathbb{N}, R), \qquad \mathbb{G}_m(R) = \mathcal{C}\mathrm{Alg}(\Sigma^\infty_+\mathbb{Z}, R).$$

We considered the case where R is an $\mathbb{E}_\infty$ algebra over $\mathbb{F}_p$ in Example 4.4. By construction, $\mathbb{A}^1(R)$ carries the structure of a strictly commutative monoid. The subspace $\mathbb{G}_m(R) \subset \mathbb{A}^1(R)$ is a collection of path components, and can be regarded as a $\mathbb{Z}$-module, i.e. deloops to a connective $H\mathbb{Z}$-module.

We can describe $\mathbb{A}^1(KU)$ as a simple example illustrating the use of Theorem 4.8. Write $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ for the profinite integers. Then $\pi_0\mathbb{A}^1(KU) = \{-1, 0, 1\}$,

$$\pi_n\mathbb{G}_m(KU) = \begin{cases} \widehat{\mathbb{Z}}, & n = 1; \\ \mathbb{Z}, & n = 2; \\ \widehat{\mathbb{Z}}/\mathbb{Z}, & n > 2 \text{ odd}; \end{cases}$$

and there are short exact sequences

$$0 \to \widehat{\mathbb{Z}}/\mathbb{Z} \to \pi_{2n+1}(\mathbb{A}^1(KU), 0) \to \prod_p \mathbb{Z}/(p^n) \to 0,$$

which are necessarily split as $\widehat{\mathbb{Z}}/\mathbb{Z}$ is injective. Here, the above product ranges over primes p, and all unspecified groups are zero.

This is computed as follows. Arithmetic fracture gives a Cartesian square

$$\begin{tikzcd} \mathbb{A}^1(KU) \arrow[r] \arrow[d] & \prod_p \mathbb{A}^1(KU_p) \arrow[d] \\ \mathbb{A}^1(KU_\mathbb{Q}) \arrow[r] & \mathbb{A}^1\left(\left(\prod_p KU_p\right)_\mathbb{Q}\right) \end{tikzcd}.$$

If R is a rational $\mathbb{E}_\infty$ ring, then $\mathbb{A}^1(R) = \Omega^\infty R$, and this determines the bottom two spaces. The claimed structure of $\mathbb{A}^1(KU)$ will follow easily by inspecting this square as soon as we understand $\mathbb{A}^1(KU_p)$. To that end we claim that $\pi_0\mathbb{A}^1(KU_p) \cong \mathbb{F}_p \subset \mathbb{Z}_p = \pi_0 KU_p$ is given by the image of the Teichmüller character, and that

$$\pi_1\mathbb{G}_m(KU_p) \cong \mathbb{Z}_p \cong \pi_2\mathbb{G}_m(KU_p), \qquad \pi_{2n+1}(\mathbb{A}^1(KU_p),0) \cong \mathbb{Z}/(p^n),$$

all other groups being zero.

One approach to computing this proceeds by showing that $\mathbb{A}^1(KU_p)$ fits into a fiber sequence

$$\mathbb{A}^1(KU_p) \to \Omega^\infty KU_p \to \Omega^\infty KU_p,$$

with second map corresponding to the KU_p-cohomology operation θ^p. As we wish to illustrate the use of Theorem 4.8, we will proceed in a different way, although the two approaches are not really any different.

The goal is to compute $\pi_*\mathbb{A}^1(KU_p) = \pi_*\mathcal{C}\mathrm{Alg}_{KU_p}(KU_p \widehat{\otimes} \Sigma^\infty_+\mathbb{N}, KU_p)$. As θ-rings,

$$\pi_0 KU_p \cong \mathbb{Z}_p, \qquad \psi(\lambda) = \lambda;$$
$$\pi_0 KU_p \otimes \Sigma^\infty_+\mathbb{N} \cong \mathbb{Z}_p[t], \qquad \psi(t) = t^p;$$

so Theorem 4.8 gives an isomorphism

$$\pi_0\mathbb{A}^1(KU_p) \cong \mathcal{R}\mathrm{ing}_{\mathbb{T}}(\mathbb{Z}_p[t],\mathbb{Z}_p) \cong \{\lambda \in \mathbb{Z}_p : \lambda^p = \lambda\}.$$

This is the image of the Teichmüller character as claimed.

To get at $\pi_*\mathbb{A}^1(KU_p)$, given some element ϕ of the above, we must compute the groups

$$H^*_{\mathbb{T}/\mathbb{Z}_p}(\mathbb{Z}_p[t], \pi_*\Omega^{2n}KU_p) \cong \mathrm{Ext}^*_\Delta(Q(\mathbb{Z}_p[t]),\omega^n)$$

for $n \geq 1$. Here, $\omega^n = \mathbb{Z}_p\{\beta^n\}$ has $\Gamma = \mathbb{Z}_p[\psi]$-module structure $\psi(\beta^n) = p^n\beta^n$, and thus $\Delta = \mathbb{Z}_p[\theta]$-module structure $\theta(\beta^n) = p^{n-1}\beta^n$.

Consider first the case where ϕ is in a path component corresponding to an element of $\mathbb{F}_p^\times$. These are all equivalent, so it is sufficient to consider the map

$$\phi\colon \mathbb{Z}_p[t] \to \mathbb{Z}_p, \qquad \phi(t) = 1.$$

With this augmentation, $Q(\mathbb{Z}_p[t]) = \mathbb{Z}_p\{s\}$ where s is the class of $t-1$, and this has Γ-module structure $\psi(s) = ps$, and thus Δ-module structure $\theta(s) = s$. The Koszul complex for $\mathrm{Ext}_\Delta(\mathbb{Z}_p\{s\},\omega^n)$ takes the form

$$p^{n-1} - 1\colon \mathbb{Z}_p \to \mathbb{Z}_p.$$

As $p^{n-1}-1$ is a unit unless $n = 1$, the only nonzero groups are

$$\mathrm{Ext}^0_\Delta(\mathbb{Z}_p\{s\},\omega) = \mathbb{Z}_p = \mathrm{Ext}^1_\Delta(\mathbb{Z}_p\{s\},\omega).$$

Theorem 4.8 then implies that $\pi_1\mathbb{G}_m(KU_p) = \mathbb{Z}_p = \pi_2\mathbb{G}_m(KU_p)$ as claimed.

Next consider the map

$$\phi\colon \mathbb{Z}_p[t] \to \mathbb{Z}_p, \qquad \phi(t) = 0.$$

With this augmentation, $Q(\mathbb{Z}_p[t]) = \mathbb{Z}_p\{t\}$ with Δ-module structure $\theta(t) = 0$, and the Koszul complex for $\mathrm{Ext}_\Delta(\mathbb{Z}_p\{t\}, \omega^n)$ takes the form

$$p^{n-1}\colon \mathbb{Z}_p \to \mathbb{Z}_p.$$

The resulting nonzero groups are

$$\mathrm{Ext}^1_\Delta(\mathbb{Z}_p\{t\}, \omega^n) = \mathbb{Z}/(p^{n-1}).$$

Theorem 4.8 implies that $\pi_{2n+1}\mathbb{A}^1(KU_p) = \mathbb{Z}/(p^n)$ as claimed. ◁

Example 4.13. A conjecture of Hopkins-Lurie [**HL13**, Conjecture 5.4.14], readily verified at height $h = 1$, in particular gives the following. Let E be the Lubin-Tate spectrum associated to the formal multiplicative group over an algebraically closed field κ of characteristic p. Let G be a finite p-group. Then

$$BG \simeq \mathcal{C}\mathrm{Alg}_E(E^{BG_+}, E).$$

By the p-adic Atiyah-Segal completion theorem [**AT69**, Proposition III.1.1], $E^1 BG = 0$ and $E^0 BG = W(\kappa) \otimes R(G)$, where $R(G)$ is the complex representation ring of G. By Theorem 4.8, we obtain a curious filtration of BG; to phrase it in terms of a fringed spectral sequence, this is

$$E_1^{2p,q} = \mathrm{Ext}^{2p-q}_{\mathbb{Z}_p[\theta]}(\mathbb{L}Q(R(G)); \omega^p) \Rightarrow \pi_q BG, \qquad d_{2r}^{2p,q}\colon E_{2r}^{2p,q} \to E_{2r}^{2(p+r),q-1},$$

where $\omega^p = \pi_{2p}E$. ◁

4.2.6. Topological André-Quillen cohomology. We now consider the $K(h)$-local topological André-Quillen homology and cohomology of E-algebras, where E is a Lubin-Tate spectrum of height h as in the preceding subsections. This is of particular interest due to work of Behrens-Rezk [**BR17**] [**BR20**], which constructs for a space X a natural comparison map $\Phi_h X \to \mathrm{TAQ}_{\mathbb{S}_{K(h)}}(\mathbb{S}_{K(h)}^{X_+})$, showing it to be an isomorphism in some nice cases. Here, Φ_h is the $K(h)$-local Bousfield-Kuhn functor [**Kuh08**]. This gives rise to a comparison map $E \mathbin{\widehat{\otimes}} \Phi_h X \to \mathrm{TAQ}_E(E^{X_+})$, again an isomorphism in some nice cases. This gives an approach to computing $E_*\Phi_h X$, which in turn gives an approach to computing $\pi_*\Phi_h X$ by descent along $\mathbb{S}_{K(h)} \to E$.

We must introduce some notation. Given a $K(h)$-local $\mathbb{E}_\infty$ ring spectrum R, and $M \in \mathcal{M}\mathrm{od}_R^{\mathrm{loc}}$, write $\widehat{\mathrm{TAQ}}^R(A; M) = L_{K(h)}\,\mathrm{TAQ}^R(A; M)$ for the $K(h)$-local André-Quillen homology of $A \in \mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,aug}}$ with coefficients in M. This can be characterized as the unique functor

$$\widehat{\mathrm{TAQ}}^R(-; M)\colon \mathcal{C}\mathrm{Alg}_R^{\mathrm{loc,aug}} \to \mathcal{M}\mathrm{od}_R^{\mathrm{loc}}$$

which preserves geometric realizations and satisfies

$$\widehat{\mathrm{TAQ}}^R(\widehat{\mathbb{P}}_R N; M) = M \mathbin{\widehat{\otimes}_R} N$$

for $N \in \mathcal{M}\mathrm{od}_R^{\mathrm{loc}}$. In addition, write $\mathrm{TAQ}_R(A; M) = \mathcal{M}\mathrm{od}_R(\widehat{\mathrm{TAQ}}^R(A, R), M)$. When $M = R$, we omit it from the notation.

On the algebraic side, given $R \in \mathcal{R}\mathrm{ing}_{\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$ and $M \in \mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m}),\heartsuit}$, define

$$\widehat{\mathbb{T}\mathrm{AQ}}^R(-; M) \colon \mathcal{R}\mathrm{ing}_{R\otimes\mathbb{T}}^{\mathrm{Cpl}(\mathfrak{m}),\mathrm{aug}} \to \mathcal{M}\mathrm{od}_R^{\mathrm{Cpl}(\mathfrak{m})}$$

to be the unique functor preserving geometric realizations and satisfying

$$\widehat{\mathbb{T}\mathrm{AQ}}^R(R \mathbin{\widehat{\otimes}^{\mathbb{L}}_{E_*}} \widehat{\mathbb{T}}(P)) = M \mathbin{\widehat{\otimes}^{\mathbb{L}}_{E_*}} P$$

for $P \in \mathcal{M}\mathrm{od}_{E_*}^{\mathrm{Cpl}(\mathfrak{m}),\mathrm{free}}$. In addition, set

$$\mathbb{T}\mathrm{AQ}_R(A; M) = \mathcal{E}\mathrm{xt}_R(\widehat{\mathbb{T}\mathrm{AQ}}^R(A; R), M).$$

Observe that

$$\widehat{\mathbb{T}\mathrm{AQ}}^R(A; M) \simeq M \mathbin{\widehat{\otimes}^{\mathbb{L}}_R} \epsilon_! \mathbb{L}\widehat{Q}_R(A) \simeq \overline{M} \mathbin{\widehat{\otimes}^{\mathbb{L}}_{R\otimes\Delta}} \mathbb{L}\widehat{Q}_R(A),$$

where $\epsilon \colon R \otimes \Delta \to R$ is the augmentation and $\epsilon_!$ should be interpreted in the derived sense, and that

$$\mathbb{T}\mathrm{AQ}_R(A; M) \simeq \mathcal{H}_{R\otimes\mathbb{T}/R}(A; \overline{M}).$$

The following theorem generalizes [**BR17**, Proposition 4.7].

THEOREM 4.10. Fix $R \in \mathcal{C}\mathrm{Alg}_E^{\mathrm{loc}}$ and $M \in \mathcal{M}\mathrm{od}_R^{\mathrm{loc}}$, and choose $A \in \mathcal{C}\mathrm{Alg}_R^{\mathrm{loc},\mathrm{aug}}$. Then there is a conditionally convergent spectral sequence of signature

$$E_1^{p,q} = \mathbb{T}\mathrm{AQ}_{R_*}^{p+q}(A_*; \omega^{-p/2} \otimes M_*) \Rightarrow \mathrm{TAQ}_R^q(A; M), \qquad d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q+1}.$$

If R_* is tame, then there is a spectral sequence of signature

$$E^1_{p,q} = \widehat{\mathbb{T}\mathrm{AQ}}^{R_*}_{p+q}(A_*; \omega^{p/2} \otimes M_*) \Rightarrow \widehat{\mathrm{TAQ}}^R_q(A; M), \qquad d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q-1},$$

which is convergent if each $\widehat{\mathbb{T}\mathrm{AQ}}^{R_*}(A_*; \omega^{p/2} \otimes M_*)$ is truncated.

PROOF. The first spectral sequence can be obtained by patching together the filtrations of $\mathcal{C}\mathrm{Alg}_R^{\mathrm{aug}}(A; R \ltimes \Sigma^n M)$ for various n given by Theorem 4.8. For the second, tameness of R guarantees that $\mathcal{M}\mathrm{odel}_{\mathrm{h}\mathcal{C}\mathrm{Alg}_R^{\mathrm{loc},\mathrm{aug},\mathrm{free}}} \simeq \mathcal{R}\mathrm{ing}^{\mathrm{Cpl}(\mathfrak{m})}_{R_*\otimes\mathbb{T}/R_*}$ and $\mathrm{L}\mathcal{M}\mathrm{od}_{\mathrm{h}\mathcal{M}\mathrm{od}_R^{\mathrm{loc},\mathrm{free}}} \simeq \mathcal{M}\mathrm{od}_{R_*}^{\mathrm{Cpl}(\mathfrak{m})}$, and the spectral sequence can be obtained as a special case of Theorem 2.7. $\square$

We expect it could be possible to remove the tameness assumption in Theorem 4.10, but we have no reason to do so. The most important case is $R = E = M$, and we will write $\mathrm{nul} = \overline{E}_* \in \mathrm{L}\mathcal{M}\mathrm{od}_\Delta$ and $\mathbb{T}\mathrm{AQ}_{E_*} = \mathbb{T}\mathrm{AQ}$.

Remark 4.12. The action of Δ on $\omega^{p/2} \otimes \overline{M}_*$ as it appears in Theorem 4.10 is through the augmentation on Δ; the presence of $\omega^{p/2}$ only serves to shift degrees. In particular, once a trivialization of ω is chosen, $\omega^{p/2} \otimes \overline{M}_* = s^{-p}\overline{M}_*$ depends only on the congruence class of p mod 2. However, these powers of ω also serve to track additional structure that is present in examples of interest, such as actions by the Morava stabilizer group. $\triangleleft$

Example 4.14. In Remark 4.11, we asserted that $\widehat{Q}(E_0^\wedge BU)$ is not projective over Δ at heights $h \geq 2$. Indeed, if it were then the spectral sequence of Theorem 4.10 would collapse, and in particular $\widehat{\mathrm{TAQ}}_E(E \mathbin{\widehat{\otimes}} \Sigma^\infty_+ BU)$ would be some E-module of infinite rank. But $\widehat{\mathrm{TAQ}}_E(E \mathbin{\widehat{\otimes}} \Sigma^\infty_+ BU) \simeq E \mathbin{\widehat{\otimes}} KU_{\geq 1} \simeq 0$ as $L_{K(h)}KU_{\geq 1} \simeq 0$ for $h \geq 2$. $\triangleleft$

In [**Rez13**] and [**Zhu18**], $E_*^\wedge \Phi_h S^{2k+1}$ is computed for $h \leq 2$; the computation proceeds by computing the cohomology groups $\mathbb{T}\mathrm{AQ}^n(E^*S^{2k+1}; \omega^{m/2} \otimes \mathrm{nul})$. In particular, it is shown in these $h \leq 2$ cases that this group vanishes for $n \neq h$. In general, let us say that E satisfies the weak algebraicity condition if $\mathbb{T}\mathrm{AQ}^n(E^*S^{2k+1}; \omega^{m/2} \otimes \mathrm{nul}) = 0$ for $n \neq h$.

Work of Bousfield [**Bou99**] [**Bou07**] describes the v_1-periodic homotopy groups of nice spaces in terms of their p-adic K-theory. One obstruction to extending this to higher heights using TAQ is in determining when $\Phi_h X \simeq \mathrm{TAQ}_{\mathbb{S}_{K(h)}}(\mathbb{S}_{K(h)}^{X_+})$; we will not consider this issue here. Provided one takes this as known, Bousfield's description of $\pi_*\Phi_1 X$ for nice spaces X at primes $p \geq 3$ can be reinterpreted as a description of $KU^\wedge_{p,*}\Phi_1 X$ that can be obtained from Theorem 4.10, combined with the standard fiber sequence $\Phi_1 X \to KU_p \mathbin{\widehat{\otimes}} \Phi_1 X \to KU_p \mathbin{\widehat{\otimes}} \Phi_1 X$. We view the following observation as the first part of a higher height analogue.

Proposition 4.9. Suppose E satisfies the weak algebraicity condition defined above. Let X be a simply connected space such that $H^*(X)$ is a finitely generated exterior algebra on odd-dimensional classes. Then

$$\mathrm{TAQ}^q_E(E^{X_+}) \cong \mathrm{Ext}^h_\Delta(Q(E^*X); \omega^{(q-h)/2} \otimes \mathrm{nul})$$

Proof. Write $H^*X \simeq \Lambda(t_1, \ldots, t_n)$ with $|t_i| = m_i$. The Atiyah-Hirzebruch spectral sequence collapses to give $E^*X \simeq \Lambda_{E_*}(x_1, \ldots, x_n)$. More precisely, there is a cell structure on X with the following property. Write $X_{\leq n}$ for the n-skeleton of X. Then the cofibering $X_{\leq n-1} \to X_{\leq n} \to \bigvee S^n$ induces a short exact sequence on H^* and E^*, and the restriction of x_i to $X_{\leq m_i}$ is the image of a generator of some $E^*S^{m_i}$. Now $\mathbb{L}Q(E^*X) = Q(E^*X) = E_*\{x_1, \ldots, x_n\}$ as E_*-modules, each $E_*\{x_i, \ldots, x_n\} \subset Q(E^*X)$ is a sub-Δ-module, and $E_*\{x_i, x_{i+1}, \ldots, x_n\}/E_*\{x_{i+1}, \ldots, x_n\} \cong \omega^{m_i/2}$. This gives a finite filtration of the Δ-module $Q(E^*X)$ with filtration quotients given by various $\omega^{m_i/2}$, and the associated spectral sequence for $\mathrm{Ext}^*_\Delta(Q(E^*X); \omega^{p/2} \otimes \mathrm{nul})$ collapses by the weak algebraicity condition, implying that it is concentrated in degree h. This implies that the spectral sequence of Theorem 4.10 collapses on a single line into the claimed isomorphism. $\square$

EXAMPLE 4.15. At heights $h \leq 2$, at least for small primes, the algebraic input to Theorem 4.8 and Theorem 4.10 is computationally accessible. As all of our examples so far have been of a general nature, let us illustrate this with a computation of $\mathrm{TAQ}^*_E(E^{SU(4)_+})$ for E the Lubin-Tate spectrum of height 2 considered in Example 4.10. There is nothing special to $SU(4)$; the method of computation may be applied more generally. There is something special to our choice of E, as we require a good understanding of how power operations act on $E^0BU(1)$.

Consider $SU(n)$ in general. This space satisfies the conditions of Proposition 4.9, and so to compute $\mathrm{TAQ}^*_E(E^{SU(n)_+})$ it is sufficient to compute $\mathrm{Ext}^2_\Delta(Q(E^*SU(n)), \omega^{1/2} \otimes \mathrm{nul})$. This is carried out in two steps; first one must understand the Δ-module $Q(E^*SU(n))$, and then one must carry out the Ext calculation. The first step might be regarded as homotopical input to the calculation, in the sense that it is not an instance of the general algebra of E-power operations. The second step is then purely algebraic. Following our discussion in Example 4.10, the image of $Q(E^*SU(n))$ under the Morita equivalence $\omega^{-1/2} \otimes -\colon \mathrm{L}\mathcal{M}\mathrm{od}_\Delta \to \mathrm{L}\mathcal{M}\mathrm{od}_\Gamma$ is determined by the Γ-module structure of $Q(E^*SU(n))$. Thus the first step reduces to understanding $Q(E^*SU(n))$ as a Γ-module, and the second step transforms into computing $\mathrm{Ext}^2_\Gamma(\omega^{-1/2} \otimes Q(E^*SU(n)), \mathrm{nul})$.

There is a suspension map $\Sigma\Sigma^\infty SU(n) \to \Sigma^\infty BSU(n)$, and this induces an isomorphism $Q(E^*SU(n)) \cong Q(E^*BSU(n))$ of Γ-modules. As $E^0BSU(n) = E_0[[c_2, c_3, \ldots]]/(c_{n+1}, \ldots)$ and $E^0BSU = E_0[[c_2, c_3, \ldots]]$ sits inside $E^0BU = E_0[[c_1, c_2, \ldots]]$ in the obvious way, one therefore reduces to computing the action of Γ on E^0BU modulo indecomposables.

Write $E^0BU(1) = E_0[[u]]$. The summation maps $BU(1)^{\times m} \to BU(m) \to BU$ induce maps $E^0BU \to E_0[[u_1, \ldots, u_m]]$ sending c_j to the j'th elementary symmetric polynomial in $u_1, \ldots, u_m$, and in the limiting case this identifies E^0BU as the ring of symmetric functions in the variables u_i. Recall from Example 4.7 the coalgebraic interpretation of Γ-modules. As mentioned at the end of Subsection 4.2.3, Schumann [**Sch14**] has computed the coaction P on $E^0BU(1)$ in closed form to be $P(u) = \frac{u^2 - du}{1 + d^2u}$. Given a symmetric function s and function f, write $s \wr f$ for the symmetric function $(s \wr f)(u_1, \ldots) = s(f(u_1), \ldots)$. Then putting everything together, the coaction on E^0BU is the map

$$P'\colon E_0[[c_1, c_2, \ldots]] \to E_0[[c_1, c_2, \ldots]][d]/(d^3 = ad + 2), \qquad P'(c_m) = c_m \wr P.$$

By Newton's identities, if $P(u)^m = \sum_j \alpha_j u^j$ with $\alpha_j \in E_0[d]/(d^3 = ad + 2)$, then P' reduces modulo indecomposables to $P'(c_m) \equiv \frac{(-1)^m}{m} \sum_j (-1)^j j \alpha_j c_j$. Finally the coaction on $\omega^{-1/2} \otimes Q(E^*SU(n)) = E_0\{c_2, \ldots, c_n\}$ is given by $P'' = \frac{-1}{d} P'$.

Specialize now to $n = 4$. Write $M = \omega^{-1/2} \otimes Q(E^*SU(4)) = E_0\{c_2, c_3, c_4\}$. As

$$\begin{aligned}
P(u)^2 &= d^2u^2 - 2(ad^2 + 3d)u^3 + (1 + 4d^3 + 3d^6)u^4 + O(u^5) \\
P(u)^3 &= -d^3u^3 + 3(d^2 + d^5)u^4 + O(u^5) \\
P(u)^4 &= d^4u^4 + O(u^5),
\end{aligned}$$

it follows that

$$\begin{aligned}
P'(c_2) &= d^2c_2 + 3(d + d^4)c_3 + 2(1 + 4d^3 + 3d^6)c_4 \\
P'(c_3) &= -d^3c_3 - 4(d^2 + d^5)c_4 \\
P'(c_4) &= d^4c_4.
\end{aligned}$$

Using $\frac{-2}{d} = a - d^2$, we divide by $-d$ to obtain

$$\begin{aligned}
P''(c_2) &= -dc_2 - 3(1 + d^3)c_3 - (d^2 - a)(1 + 4d^3 + 3d^6)c_4 \\
P''(c_3) &= d^2c_3 + 4(d + d^4)c_4 \\
P''(c_4) &= -d^3c_4.
\end{aligned}$$

Somewhat abusively, write $M^\vee = \{c_2, c_3, c_4\}$, where c_n is dual to c_n. Then the adjoint map $P^\vee\colon M^\vee \to M^\vee \otimes_t \Gamma[1]^\vee$ is given by

$$\begin{aligned}
P^\vee(c_2) &= -c_2d \\
P^\vee(c_3) &= c_3d^2 - 3c_2(1 + d^3) \\
P^\vee(c_4) &= -c_4d^3 + 4c_3(d + d^4) - c_2(d^2 - a)(1 + 4d^3 + 3d^6).
\end{aligned}$$

By Theorem 3.3 and Remark 4.7, the Koszul differential $\delta\colon K^1(M; \mathrm{nul}) \to K^2_\Gamma(M; \mathrm{nul})$ is the composite

$$\begin{tikzcd}
M^\vee \otimes_t \Gamma[1]^\vee \arrow[r, "P^\vee\otimes\Gamma[1]^\vee"] \arrow[d, dashed, "\delta"] & M^\vee \otimes_t \Gamma[1]^\vee {}_s\otimes_t \Gamma[1]^\vee \arrow[d, "M^\vee\otimes f"] \\
M^\vee \otimes_t \Gamma[1]^\vee/s(E_0) & M^\vee \otimes_t \Gamma[1]^\vee \arrow[l]
\end{tikzcd}.$$

For example, as $P^\vee(c_2) = -c_2d$, it follows that $(P^\vee \otimes \Gamma[1]^\vee)(c_2d^2) = -c_2d'd^2$; this is sent to $-c_2(a - d^2)d^2 = 2c_2d$ under $M^\vee \otimes f$, and thus $\delta(c_2d^2) = 2dc_2$.

In general, we compute δ to be given by

$$\begin{aligned}
\delta(c_2) &= c_2d^2 \\
\delta(c_2d) &= 0 \\
\delta(c_2d^2) &= c_2(2d) \\
\delta(c_3) &= c_3(2d - ad^2) + c_2(-6ad + 3a^2d^2)
\end{aligned}$$

$$
\begin{aligned}
\delta(c_3 d) &= c_3(2d^2) + c_2(9d - 6ad^2) \\
\delta(c_3 d^2) &= c_2(9d^2) \\
\delta(c_4) &= c_4(-2ad + a^2d^2) + c_3(8a^2 d + (12 - 4a^3)d^2) \\
&\quad + c_2((-66 - 40a^2 + 88a^3 + 6a^5 - 6a^6)d + (113a + 32a^3 - 56a^4 - 3a^6 + 3a^7)d^2) \\
\delta(c_4 d) &= c_4(4d - 2ad^2) + c_3(-16ad + 8a^2d^2) \\
&\quad + c_2((33a - 128a^2 - 12a^4 + 12a^5)d + (-66 - 40a^2 + 88a^3 + 6a^5 - 6a^6)d^2) \\
\delta(c_4 d^2) &= c_4(4d^2) + c_3(24d - 16ad^2) \\
&\quad + c_2((160a + 24a^3 - 24a^4)d + (33a - 128a^2 - 12a^4 + 12a^5)d^2).
\end{aligned}
$$

Here one can observe the manner in which $K_\Gamma(M; \mathrm{nul})$ is built from $K_\Gamma(\omega^n; \mathrm{nul})$ for $1 \leq n \leq 3$; compare [**Zhu18**, Example 4]. In the end, $\mathrm{TAQ}^0(E^{SU(4)_+}) = 0$, and if we write $w = c_3 d$, $x = c_3 d^2$, $y = c_4 d$, $z = c_4 d^2$, then $\mathrm{TAQ}^1(E^{SU(4)_+})$ is isomorphic to $E_0\{w, x, y, z\}$ modulo

$$
\begin{aligned}
4x &= 0 \\
2w &= ax \\
4z &= 0 \\
2ay &= a^2 z \\
4y &= 2a(x + z).
\end{aligned}
$$

By [**BR17**, Proposition 8.8], this also describes $E_*^\wedge \Phi_2 SU(4)$. $\triangleleft$

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